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Risky Human Capital and Deferred Capital Income Taxation[*]

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Abstract

We study the structure of optimal wedges and capital taxes in a Mirrlees economy with endogenous skills. Human capital is a private state variable that drives the skill process of each individual. Building on the findings of the labor literature, we assume that human capital investment is a) risky, b) made early in the life-cycle, and c) hard to distinguish from consumption. These assumptions lead to the optimality of a) a human capital premium, i.e., an excess return on human capital relative to physical capital, b) a large intertemporal wedge early in the life-cycle stemming from the lack of Rogerson's [Econometrica, 1985] "inverse Euler" characterization of the optimal consumption process, and c) an intra-temporal distortion of the effort/consumption margin even at the top of the skill distribution at all dates except the terminal date. The main implication for the structure of linear capital taxes is the necessity of deferred taxation of physical capital. In particular, deferred taxation of capital prevents the agents from making a joint deviation of under-investing in human capital ex ante and shirking from labor effort at some future date in the life-cycle, as the marginal deferred tax rate on physical capital held early in the life-cycle is history-dependent. The average marginal tax rate on physical capital held in every period is zero in present value. Thus, as in Kocherlakota [Econometrica, 2005], the government revenue from capital taxation is zero. However, since a portion of the capital tax must be deferred, expected capital tax payments cannot be zero in every period. Necessarily, agents face negative expected capital tax payments due early in the life-cycle and positive expected capital tax payments late in the life-cycle. Also, relative to economies with exogenous skills, the optimal marginal wealth tax rate is more volatile.

Keywords: Optimal taxation, private information, human capital, deferred tax.
JEL classification: E62, H21, J24.

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1 Introduction

Recent literature obtains important results characterizing optimal capital and labor income taxes in dynamic Mirrlees economies.[1] In a Mirrlees economy, agents are affected by idiosyncractic, privately observable shocks to the productivity of their labor effort. In the Mirrlees approach to optimal taxation, the role of the tax system is to fund government purchases and insure the productivity risk. The optimal taxation problem is to characterize a tax system that fulfills this dual role efficiently, given the informational constraints imposed by the lack of public observability of the idiosyncractic productivity shocks.

In a ground-breaking paper, Mirrlees (1971) solves the optimal taxation problem in a static setting. Taking all but the labor effort decisions as given, he characterizes optimal labor income taxes. The main limitation of the static approach is that it ignores the effect that taxes have on agents' investment decisions. The contribution of the recent literature is in the characterization of optimal tax systems in dynamic settings in which agents' physical capital investment decisions (i.e., savings) are endogenous.

Physical capital investment, however, is not the only important category of investment decision problems that agents face over the life-cycle. There is ample evidence suggesting that human capital investment decisions are at the very least equally important.[2] By investing in their human capital, people affect profoundly their future skills, and, consequently, their wages, earnings, and welfare. The existing literature on optimal taxation with endogenous savings, however, takes the evolution of agents' skills as exogenous and thus ignores the effect that taxes have on agents' human capital investment decisions. In this paper, we solve the optimal taxation problem in a dynamic setting in which both the physical capital and human capital investment decisions are endogenous.

Given that effort is not observable, human capital, defined as an individual-specific state variable that determines the productivity labor effort, is not directly observed in the data. There is, however, a large empirical literature on human capital, which identifies a host of important properties of the process of human capital formation and evolution over the life-cycle. From the vantage point of the Mirrlees model, three of these properties come to the forefront of importance.

First, recent studies summarized in Carnerio and Heckman (2003) document that most of human capital investment is done early in the life-cycle. We incorporate this fact in our model by assuming that human capital investment is undertaken only at the first date in the agent's life-cycle.

Second, it has been well documented in empirical studies that the returns on human capital investments are risky.[3] We capture this fact in our model by assuming that initial human capital

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[1] See Kocherlakota (2006) for a review.

[2] Some estimates put the value of human capital at 93% of all wealth in the US. See Palacios-Huerta (2003a) and references therein.

[3] See Palacios-Huerta (2003) who documents that the variation in the stochastic properties of different human

investment is subject to a stochastic productivity shock, and the level of accumulated human capital is subject to stochastic depreciation shocks throughout the agent's life-cycle.

Third, the economics literature has long recognized the difficulty in distinguishing human capital investment from ordinary consumption expenditure.[4] It has also been recognized as a problem in the ongoing policy debate on how to design the tax system in order to foster human capital accumulation.[5] At the core of this measurement problem lies the fact that, in reality, there is a human capital investment component in ordinary consumption and a significant amount of consumption value in human capital investment activities such as education and training. Agents use a large variety of goods, services, and nonmarket activities as vehicles for their human capital investment and consumption. It is difficult to measure the relative "loadings" of human capital investment and pure consumption embedded in a particular good or service. In order to capture this measurement problem in a model with a single consumption good, we assume that consumption and human capital investment are indistinguishable to an outside observer.

In this paper, we introduce endogenous human capital into a life-cycle Mirrlees economy, taking into account all three of the aforementioned properties of the human capital accumulation process.[6] We characterize the optimal allocation of labor, consumption, physical and human capital investment, and construct a tax system that implements this allocation in equilibrium. We derive two kinds of results. First, we obtain a set of results about the structure of optimal wedges in our environment.[7] Then, we derive our main results concerning the structure of optimal capital taxes.

We characterize three types of wedges: the intra- and inter-temporal wedges, which have their counterparts in the existing literature on dynamic Mirrlees economies, plus an additional wedge that has not been identified before, which represents the optimality of a human capital premium.

In the existing literature, the optimality of the intertemporal wedge, i.e., the inequality between

capital returns is substantial, especially when comparing human capital and liquid assets. This fact is also supported by the empirical studies showing that much of the variation in individual or household earnings in US panel data is not explained by individual variables such as age, sex, education, or by aggregate variables (See, e.g., Meghir and Pistaferri (2004), and Storesletten et al (2004)).

[4] As early as in 1961, Theodore Schultz in his Presidential Address to the AEA raises this question by saying "How can we estimate the magnitude of human capital investment? [...] Most relevant activities clearly are [...] partly consumption and partly investment, which is why the task of identifying each component is so formidable." See Schultz, (1961, 1961a) and Shaffer (1961) for an extensive discussion. Also, see Heckman (1976), Heckman (1999), Davies, Zeng and Zhang (2000), Carneiro and Heckman (2003).

[5] A 2005 memorandum to the President's Advisory Panel on Federal Tax Reform on tax treatment of investment in human capital prepared by the Treasury Department's Office of Tax Analysis says, "In practice, it can be very difficult to distinguish between human capital investment and education consumption." See the reference United States Department of Treasury, Office of Tax Analysis (2005) for a full discussion.

[6] Many microeconomic studies also include effort as an input in the technology of human capital production. We do not include this input in our production function for clarity of exposition. Our results do not depend on this abstraction.

[7] In the public finance literature, the difference between a marginal rate of substitution and a corresponding marginal technical rate of transformation is known as a wedge. The importance of wedges associated with a given allocation comes from the difficulty that they present for equilibrium implementation. If agents' access to the available technology is unrestricted, i.e., undistorted by taxes, no allocation with non-zero wedges can be implemented in equilibrium. Thus, wedges determine taxes; see Kocherlakota (2004).

agents' shadow interest rates and the marginal rate of transformation of consumption across time, follows from the martingale property of the discounted inverse marginal utility of consumption. This property of the optimal allocation, which is often referred to as the inverse Euler equation or the Rogerson condition [see Rogerson (1985), and Kocherlakota, Golosov and Tsyvinski (2003)], implies that agents are savings-constrained at the optimum, i.e., individual shadow interest rates are strictly greater than the rate of return on savings. Similarly, in our economy with endogenous human capital, agents are savings-constrained at the optimum. The martingale property, however, does not hold in every period. In particular, early in the life-cycle when private human capital investment is made, the discounted inverse marginal utility of consumption is a strict supermartingale. This effect reinforces the intertemporal wedge in our environment relative to the environments in which skills are exogenous.

The discrepancies between agents' willingness to substitute leisure for consumption and the marginal rate of transformation (the wage rate) are known as intratemporal wedges. The structure of optimal intratemporal wedges obtained in our environment is different from those obtained in most Mirrlees economies. The usual no-distortion-at-the-top property does not hold at the optimal allocation of our economy. In particular, we find that, at all non-terminal dates in the life-cycle and for all agents at the top of the cross-sectional distribution of individual productivity, the marginal utility of an additional unit of consumption is strictly lower than the marginal disutility of effort necessary to produce this unit.

Given a fixed pattern of labor effort, each additional unit of human capital investment generates a gain in the expected future output. This gain provides a measure of return on human capital investment that is directly comparable with the rate of return on physical capital investment. We find that, at the optimum, the return on human capital investment exceeds the return on physical capital investment, which demonstrates the optimality of a human capital premium. This difference between the rates of return on the two types of capital constitutes an additional wedge, which needs to be accounted for in a market implementation. This wedge, which we term *asset return wedge*, is new to the literature on Mirrlees-type economies.

The main results of the paper provide a characterization of a tax system that implements the optimum. We study a standard market equilibrium model in which agents freely trade capital and labor subject to taxes. We follow Kocherlakota (2005) in our focus on tax systems that are fully history-dependent, nonlinear in labor income, and linear in capital. Our main result is the necessity of deferred taxation of capital. We demonstrate that if capital can only be taxed contemporaneously, then implementation of the optimal allocation in a market equilibrium is impossible. Then, we show that there exists an optimal tax system in which taxes on capital held early in the life cycle are deferred until later in the life-cycle, when all individual uncertainty has been resolved. The amount

of deferred tax obligation is linear in capital saved during the initial period, when human capital investment is made. The key finding is that the marginal tax rate that determines the deferred capital tax obligation has to be history-dependent. In particular, high deferred taxes are levied on agents with low labor income profiles, and those with high labor income profiles pay low deferred capital taxes.

Intuition about these results comes from the incentive problem that determines the optimal allocation and, consequently, shapes optimal tax structures in our environment. In Mirrlees economies with exogenous skills [Kocherlakota (2005), Albanesi and Sleet (2006)], taxes, in addition to raising revenue, must provide incentives to prevent high-skilled agents from pretending to be low-skilled, i.e., from shirking. Savings and shirking are complementary: increasing one's savings makes shirking more attractive. Capital taxes, thus, are high for agents whose labor income is low, as low labor income is consistent with shirking. In our model, agents can end up highly productive ex post only if they make sufficient human capital investment ex ante. Taxes, therefore, must provide enough incentives not only to discourage shirking throughout the life-cycle, but also to encourage sufficient investment in human capital at the beginning of the life-cycle. Agents' human capital investment and labor effort choices are private and complementary: if an agent plans to shirk, underinvesting in human capital increases the value of shirking. This value is further increased by over-saving. Similar to the exogenous skill case, high capital taxes on agents with low labor income (suspected shirkers) eliminate this complementarity. However, due to the dynamic nature of agents' deviation strategies (in which agents under-invest in human capital and over-save early, and shirk later in the life-cycle), partial labor income histories in general do not carry all the information needed for the tax system to efficiently deter these joint deviations.

As an example, consider a deviation plan that does not call for shirking until, say, age 40. Agents who follow this deviation plan work hard in their 20s and 30s producing labor income profiles that are identical to those produced by agents who do not plan to shirk at all. During this period, thus, observed labor income profiles contain no indication of deviation from the optimal behavior. Those who plan to shirk at age 40, however, want to under-invest in human capital and over-save early on, say, already at age 25. Low labor income at 40 is the first indication of their deviation, and this information has to be used to deter early over-saving. This is achieved by applying a high marginal tax rate at 40 to savings held at 25. In contrast, those with high labor income at 40 pay low deferred capital taxes on their savings held at age 25, as there is still no reason to suspect that these agents are shirking. If labor income turns out to be low later on, say at 50, a high deferred capital tax must be applied then in order to deter the strategy of shirking at 50. However, the deferred tax hit is not as big as it is for those who produce low labor incomes at 40 because shirking at 50 is not, in a sense, as socially damaging as shirking already at the age of 40.

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Our result regarding the efficiency of deferred capital taxation is similar to the point about estate taxation made recently in Kopczuk (2003). This paper points out that when annuity markets are imperfect, there is an efficiency advantage of estate taxation over income taxation stemming from the insurance against the longevity risk that estate taxation provides by deferring taxes until all uncertainty about the length of the lifespan has been resolved. Those who live long, run down their assets and end up paying low estate taxes. Those who die early, leave a lot of assets to their estates, which results in high estate tax obligations. The efficiency of the estate tax, viewed as a deferred income tax, comes from the fact that it makes use of more information than (contemporaneous) income taxes do, as the realized length of the lifespan is not known at the time when income is earned. Similarly, in our model, deferring capital taxes is efficient because with more observations of agents' labor incomes becoming available, more information is revealed about agents' private human capital investments. This additional information is used to efficiently provide incentives for agents to invest in human capital and exert labor effort.

The proof of our implementation result is constructive, so we can provide a full characterization of an optimal tax system. In particular, we show that, similar to Kocherlakota (2005), it is optimal for the government revenue from taxation of capital to be zero. In our environment, this does not imply, however, that expected capital taxes are zero in every period. Since deferred taxes are necessary in our model, agents face negative expected capital taxes early in the life-cycle and positive expected capital taxes later in the life-cycle. The ex ante expected present value of lifetime capital taxes paid by every agent is zero, which implies that the present value of the government revenue from taxation of capital is zero. This result is intuitive. There is no reason for the government to raise revenue via distortionary capital taxation when lump-sum and nondistortionary (fully nonlinear) labor income taxes are available. Under the optimal system, capital taxes are used purely to provide correct incentives to the agents. All redistribution and social insurance transfers are implemented via nondistortionary lump-sum and labor income taxes.

The last result we present concerns the larger volatility of marginal tax rates needed for the implementation of a given optimal allocation in our endogenous-skill environment relative to a similar environment in which skills are exogenous. This result follows from comparing the structures of the incentive problems in the exogenous- and endogenous-skill models. Allowing for endogenous human capital accumulation through unobservable investment adds in our environment an extra dimension to the space of strategies that agents can use to deviate from the socially optimal pattern of behavior. With such an enhanced set of deviation opportunities, the incentive problem of our environment is more severe, relative to environments in which skills are exogenous. This translates, at the optimum, into a larger intertemporal wedge between the shadow interest rate of consumption and the rate of return on physical capital investment early in the life-cycle when human capital

investment is made. In order to support this wedge in equilibrium, capital taxes have to introduce more risk into the return on physical capital investment, which means that the present value of ex post marginal capital tax rates has to be more volatile when skills are endogenous.

The question of optimal taxation in a model with human capital accumulation has been addressed in many papers in the context of the so-called Ramsey approach to optimal taxation [see for example Jones, Manuelli and Rossi (1997)], in which the government is restricted to use proportional taxes. Our paper is different, as we use the Mirrlees approach.

Our paper is closely related to Kocherlakota (2005). This paper characterizes an optimal system of linear capital taxes in a very rich economic environment with exogenous skills. Deferred taxes are unnecessary in that environment. Our paper shows that deferred capital taxes become necessary when skill formation is introduced into the Mirrlees model under assumptions consistent with three basic empirical facts about the skill formation process. As Kocherlakota (2005) points out, his analysis can be extended, without affecting his results, to an endogenous skills economy in which human capital investment is separable from consumption. However, this extension would not capture the important fact of nonseparability between human capital investment and consumption, which we capture in our model.

Kapicka (2006) studies optimal taxation in a Mirrlees environment with human capital. There are several important features of that model that differentiate it from our paper, which include the following. Human capital investment is assumed to be riskless and separable from consumption. There is no physical capital and all intertemporal trade is shut down: neither the agents nor the government have access to markets for claims to future consumption. Also, the government cannot keep record of agents' past income, and so labor income taxes are restricted to depend on current income only.

Farhi and Werning (2006) study optimal estate taxation in a dynamic Mirrlees environment with exogenous skills, where the effective social discount rate is lower than the private one. In their environment, the optimal allocation does not satisfy the Rogerson condition and the average optimal linear capital (i.e., estate) tax rate is not zero. Albanesi (2006) shows that, in a model with entrepreneurial capital and moral hazard where the optimal allocation does satisfy the Rogerson condition, the average optimal marginal tax rates on all assets are zero. The results of Farhi and Werning (2006) and Albanesi (2006), together with the original result of Kocherlakota (2005), suggest that, in a class of tax systems that are linear in capital, the zero average capital tax result holds if and only if the Rogerson condition is satisfied at the optimum. Our results contradict this intuition. In the environment we study, the optimal allocation does not satisfy the Rogerson condition but the present value of expected marginal capital tax rate is zero. Albanesi and Sleet (2006) show that capital taxes may be nonzero even in environments in which the Rogerson condition holds at

the optimum. Their tax system, however, is generally nonlinear in wealth and, thus, their results concern a different decentralization.

The rest of this paper is organized as follows. Section 2 defines the environment. In section 3, we provide a characterization of the optimal allocation. In section 4, we show the existence of a tax system implementing the optimum. There, also, we provide our mail characterization results of an optimal tax system. Section 5 provides a numerical example. Section 6 concludes.

2 Environment

Consider a T-period $(t = 0, 1, ..., T)$ economy populated by a continuum of ex ante identical agents. The size of the population is normalized to unity. There is a single consumption good, a single physical capital good, and labor input measured in effective units. The initial endowment of resources consists of $K_0 > 0$ units of physical capital.

Preferences: Agents' preferences over stochastic streams of consumption $c = (c_0, c_1, ..., c_T)$ and labor effort $l = (l_1, ..., l_T)$ are represented by expected utility function

$$u_0(c_0) + E \sum_{t=1}^{T} \beta^t \left\{ u(c_t) + v(l_t) \right\}, \tag{1}$$

where $u_0 : \mathbb{R}_+ \to \mathbb{R}$ and $u : \mathbb{R}_+ \to \mathbb{R}$ are strictly increasing, strictly concave C^2 functions, u exhibits non-increasing absolute risk aversion (NIARA), and $v : \mathbb{R}_+ \to \mathbb{R}$ is a strictly decreasing, strictly concave C^2 function such that $v(0) = 0$.

Technology: At the initial date $t = 0$, physical capital K_0 can be consumed, invested in tomorrow's physical capital, K_1, or turned into human capital investment, i. The human capital production technology is stochastic. A date-zero human capital investment of size $i \geq 0$ produces the amount $h_1 \geq 0$ of date-one human capital according to the following human capital production function

$$h_1 = \theta i, \tag{2}$$

where $\theta \in \Theta \equiv \{0, 1\}$ is a shock to human capital investment productivity. The probability of the realization θ of the human capital investment shock is denoted by $\pi_0(\theta)$, with $0 < \pi_0(\theta) < 1$ for both $\theta \in \Theta$. The realizations of θ are drawn from the distribution π_0 independently for each agent. Throughout, we assume that the exact Law of Large Numbers applies, so $\pi_0(\theta)$ also represents the fraction of agents whose shock realization is θ.

At dates $t = 1, ..., T$, physical capital K_t and aggregate effective labor Y_t are used to produce output

$$F(K_t, Y_t) = Z(K_t, Y_t) + (1 - \delta)K_t,$$

which can be consumed or invested in next period's physical capital, K_{t+1}. We adopt the standard assumptions about the aggregate production function Z and the depreciation rate δ.[8]

There are no human capital investment opportunities at dates $t = 1, ..., T$. However, the initial individual human capital h_1 persists, subject to stochastic depreciation shocks. In particular, individual human capital h_t at dates $t = 2, ..., T$ is given by

$$h_t = \sigma_{t-1} h_{t-1}, \tag{3}$$

where $\sigma_t \in \Theta$ for $t = 1, ..., T - 1$ is an individual-specific human capital depreciation shock. After any partial history of individual shocks $\eta^t = (\theta, \sigma_1, ..., \sigma_{t-1}) \in \Theta^t$, the conditional probability distribution of σ_t is denoted by $\pi_t(\sigma_t | \eta^t)$. We assume that the realizations of σ_t are drawn independently for each agent in the population. Let $\pi^t : \Theta^t \to [0, 1]$ denote the probability of history η^t constructed from the probability distribution π_0 and the conditional probability distributions π_t.

Human capital determines agents' productivity. At each date $t = 1, ..., T$, an agent whose human capital is h_t and whose labor effort is l_t provides y_t units of effective labor given by

$$y_t = \psi_t(h_t) l_t,$$

where $\psi_t : \mathbb{R}_+ \to \mathbb{R}_{++}$ for each $t = 1, ..., T$ is a strictly increasing, strictly concave, differentiable human capital productivity function satisfying the Inada conditions: $\lim_{h \to 0} \psi_t'(h) = +\infty$, $\lim_{h \to \infty} \psi_t'(h) = 0$.[9] The human capital productivity functions $\{\psi_t\}_{t=1}^T$ map human capital to skill. The value $\psi_t(h_t)$ represents the skill level of an agent whose human capital in period t is h_t. Skill relates labor effort to effective labor units: one unit of effort l_t of an agent with human capital h_t produces $\psi_t(h_t)$ units of effective labor at date t. Note that, since $\psi_t(0) > 0$ for all $t = 1, ..., T$, the agents with the low human capital level $h_t = 0$ are not unable to work. Also, note that, for all $t = 1, ..., T$, the skill level at t is increasing in the initial human capital investment i.

There are two important implications of our assumptions for the evolution of the human capital process in the model. First, the low human capital state is absorbing: (3) implies that if $h_{t-1} = 0$, then $h_t = \sigma_{t-1} 0 = 0$ independently of the realization of $\sigma_{t-1} \in \Theta$. Thus, it is without loss of generality to assume that the conditional probability $\pi_t(1 | \eta^t)$ is zero for all t-element partial histories of individual shocks η^t different from the history $1^t \equiv (1, 1, ..., 1)$. Under this assumption, in order

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[8] In particular, we assume that $Z : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, strictly concave, constant returns to scale, and C^2. Also, we assume that $\delta \in [0, 1]$, and that Z satisfies the following Inada conditions:

$$\lim_{K \to 0} Z_1(K, Y) = \infty, \quad \lim_{Y \to 0} Z_2(K, Y) = \infty,$$

$$Z(K, 0) = Z(0, Y) = 0,$$

where Z_i denotes the first partial derivative of Z with respect to the i-th argument.

[9] If $\psi_t(0) = 0$ for some t, all of our results go through with minor changes.

to simplify notation, we will write $\pi_t(\sigma_t)$ as a shorthand for $\pi_t(\sigma_t|1^t)$. Although our results do not essentially depend on this, we will assume that $\pi_t(0) > 0$ for all $t = 1, ..., T-1$, i.e., the agents whose current human capital h_t is strictly positive face the risk of human capital depreciation in every period.

The second implication of the multiplicative specification (3) is that a low realization of either the human capital investment shock or any of the depreciation shocks erases all effects of the initial human capital investment level. Indeed, noting that $h_t = i\theta\sigma_1...\sigma_{t-1}$, we have that if any of the shocks θ, σ_s for $s \leq t-1$ is realized at zero, then $h_t = 0$ for all values of i.

These two properties of the human capital process significantly simplify the analysis of our model. However, given the flexibility provided by the human capital productivity functions $\{\psi_t\}_{t=1}^T$ and the conditional probability distributions $\{\pi_t\}_{t=1}^T$, the model remains general enough to admit a large class of skill processes $\{\psi_t(h_t)\}_{t=1}^T$. In particular, despite our assumption that human capital can only decrease after the initial investment period, the life-cycle sample paths of the skill process can be increasing because $\psi_t \neq \psi_s$ for $t \neq s$. This flexible specification of the human capital productivity functions $\{\psi_t\}_{t=1}^T$ makes the model consistent with a large variety of life-cycle skill profiles.

The following two definitions formally state what in this economy constitutes, respectively, an allocation, and a resource feasible allocation.

Definition 1 *A (type-identical) allocation is a collection $A = (i, h, c, l, y, K, Y)$ where*
$i \in \mathbb{R}_+$ *denotes the human capital investment level;*
$h = (h_1, ..., h_T)$ *denotes the human capital process with $h_t : \mathbb{R}_+ \times \Theta^t \to \{0, i\}$ for $t = 1, ...T$;*
$c = (c_0, c_1, ..., c_T)$ *denotes the consumption process with $c_0 \in \mathbb{R}_+$ and $c_t : \Theta^t \to \mathbb{R}_+$ for $t = 1, ...T$;*
$l = (l_1, ..., l_T)$ *denotes the labor effort process with $l_t : \Theta^t \to \mathbb{R}_+$ for $t = 1, ...T$;*
$y = (y_1, ..., y_T)$ *denotes the effective labor input process with $y_t : \Theta^t \to \mathbb{R}_+$ for $t = 1, ...T$;*
$K = (K_0, K_1, ..., K_T) \in \mathbb{R}_+^{T+1}$ *denotes the aggregate physical capital sequence, with the initial capital $K_0 > 0$ exogenously given;*
$Y = (Y_1, ..., Y_T) \in \mathbb{R}_+^T$ *denotes the aggregate effective labor input sequence.*

Definition 2 *Given the initial physical capital endowment K_0 and an exogenous sequence of government revenue $\{G_t\}_{t=0}^T$, an allocation A is resource feasible (RF) if*

$$c_0 + i + K_1 \leq K_0 - G_0,$$

$$\sum_{\eta^t \in \Theta^t} \pi^t(\eta^t)c_t(\eta^t) + K_{t+1} \leq F(K_t, Y_t) - G_t, \quad t = 1, ...T,$$

$$y_t(\eta^t) = \psi_t(h_t(\eta^t))l_t(\eta^t), \quad \eta^t \in \Theta^t, t = 1, ...T,$$

$$Y_t = \sum_{\eta^t \in \Theta^t} \pi^t(\eta^t)y_t(\eta^t), \quad t = 1, ...T.$$

Information: In this environment, publicly observable are the aggregate physical capital $\{K_t\}_{t=0}^{T}$, and each agent's individual effective labor input $\{y_t\}_{t=0}^{T}$. The individual human capital investment i, stock of human capital $\{h_t\}_{t=1}^{T}$, labor effort $\{l_t\}_{t=1}^{T}$, and all individual shocks θ, $\{\sigma_t\}_{t=1}^{T-1}$ are private information of each agent.

Individual consumption $\{c_t\}_{t=0}^{T}$ is not publicly observable. However, since savings (physical capital accumulation) are publicly observable, the actual consumption must be equal to the allocated consumption c_t at dates $t = 1, ..., T$. At these dates, any attempt by an agent to alter the assigned consumption profile (i.e., to save or borrow) would be observable. Hence, at dates $t = 1, ..., T$ consumption is effectively observable.

A key feature of our model is that the same is not true about consumption at $t = 0$. At this date, agents consume and make their human capital investment, both of which are not publicly observable. Under an allocation A, each agent receives i units of physical resources with the recommendation to invest, and c_0 units with the recommendation to consume. Agents can, however, deviate from the recommendation c_0 without being detected. They can, for example, consume more than c_0 and invest less than i without being detected, as long as the total of their actual consumption and investment adds up to $c_0 + i$. Thus, the actual consumption at $t = 0$ and human capital investment remain private information of each agent.

Due to the presence of private information, we confine attention to allocations that are incentive compatible. By the Revelation Principle, restricting attention to IC allocations is without loss of generality.[10]

Definition 3 *An allocation A is incentive compatible (IC) if*

$$u_0(c_0) + E\sum_{t=1}^{T} \beta^t \{u(c_t) + v(l_t)\}$$

$$\geq \max_{0 \leq j \leq c_0+i} u_0(c_0 + i - j) + \beta \sum_{\theta \in \Theta} \pi_0(\theta) \max_{\hat{\theta} \in \Theta} \left\{ w_1(j, \theta, \hat{\theta}) \right\}, \tag{4}$$

where

$$
\begin{aligned}
w_1(j, \theta, \hat{\theta}) &= u(c_1(\hat{\theta})) + v\left(\frac{\psi_1(i\theta) l_1(\hat{\theta})}{\psi_1(j\theta)} \right) \\
&\quad + \beta \sum_{\sigma_1 \in \Theta} \pi_1(\sigma_1 | \theta) \max_{\hat{\sigma}_1 \in \Sigma_2(\hat{\theta})} \left\{ w_2(j, (\theta, \sigma_1), (\hat{\theta}, \hat{\sigma}_1)) \right\},
\end{aligned}
$$

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[10] Formally, an incentive compatible allocation is an outcome of an incentive compatible direct revelation mechanism. In contrast, fiscal mechanisms, which we introduce in section 4, are indirect.

and for $t = 2, ..., T$

$$
\begin{aligned}
w_t(j, \eta^t, \hat{\eta}^t) &= u(c_t(\hat{\eta}^t)) + v\left(\frac{\psi_t(i1(\hat{\eta}^t))l_t(\hat{\eta}^t)}{\psi_t(j1(\eta^t))}\right) \\
&\quad + \beta \sum_{\sigma_t \in \Theta} \pi_t(\sigma_t|\eta^t) \max_{\hat{\sigma}_t \in \Sigma_{t+1}(\hat{\eta}^t)} \left\{w_{t+1}(j, (\eta^t, \sigma_t), (\hat{\eta}^t, \hat{\sigma}_t))\right\},
\end{aligned}
$$

with

$$w_{T+1} = 0,$$

where $1(\eta^t) = \theta\sigma_1...\sigma_{t-1},$ *and*

$$
\Sigma_{t+1}(\hat{\eta}^t) = \begin{cases} \{0,1\} & \text{if} \quad 1(\hat{\eta}^t) = 1, \\ \{0\} & \text{if} \quad 1(\hat{\eta}^t) = 0. \end{cases}
$$

In the above definition, $w_t(j, \eta^t, \hat{\eta}^t)$ represents the date-t continuation value of an agent whose initial human capital investment level was j and whose true and announced history of shocks up to t are, respectively, η^t and $\hat{\eta}^t$. The set $\Sigma_{t+1}(\hat{\eta}^t)$ describes the feasible reports of the shock σ_t, given the announced history $\hat{\eta}^t$ (since the low state is absorbing, once the low realization of a shock to the human capital has been reported, the high realization cannot be declared).

Allocation A satisfies the IC constraint (4) if it is individually optimal for the agents to follow the recommendation on human capital investment i at $t = 0$, and then truthfully report their individual realizations of the human capital shocks throughout the life-cycle. The left-hand side of (4) represents the value that this strategy (to be called the truthful strategy) delivers to the agents under allocation A. The right-hand side of (4) represents the maximal value that an agent can attain under A with any state-contingent, individually feasible strategy for human capital investment and shock announcement. The individually feasible strategies comprise all deviations from the truthful strategy that are, at every stage in the life-cycle, measurable with respect to the agent's information, and undetectable (i.e., impossible to distinguish from the truthful strategy) with public information.

Definition 4 *An allocation A is constrained optimal if it is incentive compatible and resource feasible and if it maximizes, in the class of all IC and RF allocations, the ex ante expected utility of the representative agent.*

By the above definition, an optimal allocation is a solution to the following *social planning problem:*

$$\max_A \ u_0(c_0) + E \sum_{t=1}^{T} \beta^t \left\{u(c_t) + v(l_t)\right\}, \tag{P1}$$

subject to (RF), (IC).

3 Characterization of the optimal allocation

In this section, we provide a characterization of the set of optimal allocations. In the first subsection, we define a relaxed social planning problem and show that, in a generic class of economies, it is a concave maximization problem whose unique solution is feasible in the (unrelaxed) social planning problem, which implies the existence of a unique optimal allocation. In the second subsection, we turn to the properties of the optimum. We provide results about intratemporal, intertemporal, and asset return wedges.

3.1 Simplifying the social planning problem

The IC constraint (4) involves on- and off-equilibrium continuation value functions $\{w_t\}_{t=1}^T$, which are complicated history-dependent objects whose properties are unknown. The following lemma simplifies this complicated IC condition by expressing it, in an equivalent form, as $2T$ inequality and 1 equality conditions. The set of off-equilibrium objects involved in this expression is reduced to T numbers $\{j_t\}_{t=1}^T$, which represent off-equilibrium human capital investment levels, and are much easier to characterize than the continuation value functions $\{w_t\}_{t=1}^T$.

Lemma 1 *An allocation A is incentive compatible if and only if the following three sets of conditions hold:*

1. *conditions $(IC_{0,t})$ for $t = 1, .., T$:*

$$u\left(c_t(1^{t-1}, 0)\right) + v\left(l_t(1^{t-1}, 0)\right) + \sum_{s=t+1}^T \beta^s \left\{u\left(c_s(1^{t-1}, 0, 0^{s-t})\right) + v\left(l_s(1^{t-1}, 0, 0^{s-t})\right)\right\}$$

$$\geq u\left(c_t(1^{t-1}, 1)\right) + v\left(\frac{\psi_t(i)l_t(1^{t-1}, 1)}{\psi_t(0)}\right) + \sum_{s=t+1}^T \beta^s \left\{u\left(c_s(1^{t-1}, 1, 0^{s-t})\right) + v\left(l_s(1^{t-1}, 1, 0^{s-t})\right)\right\},$$

where $0^t = (0, ..., 0) \in \Theta^t$ and $1^t = (1, ..., 1) \in \Theta^t$ for $t = 1, ..., T$ denote the constant partial histories of shocks to human capital, and where 0^0 and 1^0 both denote the empty history;

2. *conditions* $(IC_{1,t})$ *for* $t = 1, .., T$:

$$u_0(c_0) + \sum_{s=1}^{t-1} \beta^s \pi^s(1^s) v(l_s(1^s))$$

$$+ \beta^t \pi^t(1^t) \left\{ u\left(c_t(1^t)\right) + v(l_t(1^t)) \right\}$$

$$+ \sum_{s=t+1}^{T} \beta^s \sum_{\eta^{s-t}} \pi^s(1^t, \eta^{s-t}) \left\{ u(c_s(1^t, \eta^{s-t})) + v(l_s(1^t, \eta^{s-t})) \right\}$$

$$\geq u_0(c_0 + i - j_t) + \sum_{s=1}^{t-1} \beta^s \pi^s(1^s) v \left(\frac{\psi_s(i) l_s(1^s)}{\psi_s(j_t)} \right)$$

$$+ \beta^t \pi^t(1^t) \left\{ u\left(c_t(1^{t-1}, 0)\right) + v \left(\frac{\psi_t(0) l_t(1^{t-1}, 0)}{\psi_t(j_t)} \right) \right\}$$

$$+ \sum_{s=t+1}^{T} \beta^s \sum_{\eta^{s-t}} \pi^s(1^t, \eta^{s-t}) \left\{ u(c_s(1^{t-1}, 0, 0^{s-t})) + v \left(\frac{\psi_s(0) l_s(1^{t-1}, 0, 0^{s-t})}{\psi_s\left(j_t 1(\eta^{s-t})\right)} \right) \right\}, \quad (5)$$

where j_t *solves*

$$\frac{d}{dj} \left[u_0(c_0 + i - j) + \sum_{s=1}^{t-1} \beta^s \pi^s(1^s) v \left(\frac{\psi_s(i) l_s(1^s)}{\psi_s(j)} \right) + \sum_{s=t}^{T} \beta^s \pi^s(1^s) v \left(\frac{\psi_s(0) l_s(1^{t-1}, 0^{s-t+1})}{\psi_s(j)} \right) \right] = 0; \quad (6)$$

3. *and the single condition* (IC_i): i *solves the following equation*

$$\frac{d}{dj} \left[u_0(c_0 + i - j) + \sum_{t=1}^{T} \beta^t \pi^t(1^t) v \left(\frac{\psi_t(i) l_t(1^t)}{\psi_t(j)} \right) \right] = 0. \quad (7)$$

Proof In Appendix. ∎

By definition, an allocation is incentive compatible in our environment if investing the recommended amount and truth-telling throughout does not provide less utility to an individual agent than any available plan of deviation from the truthful strategy. In our environment, the set of deviation plans that cannot be publicly detected is quite large, as agents can privately choose a human capital investment level j, which is a continuous choice variable, and a measurable shock reporting strategy $\hat{\eta}^T : \Theta^T \to \Theta^T$. The large number of available deviation strategies makes checking for incentive compatibility of an allocation a complicated task.

The content of Lemma 1 is that when we check for incentive compatibility of an allocation, we can ignore all but $2T + 1$ of these deviation strategies. The constraint $IC_{0,t}$ for $t = 1, ..., T$ corresponds to the deviation strategy of a one-period overstatement of the skill level at date t. The constraint $IC_{1,t}$ for $t = 1, ..., T$ corresponds to the deviation strategy of shirking from date t on, combined with a deviation in the human capital investment level at $t = 0$. At this deviation, the human capital investment level, denoted by j_t, is the amount of human capital investment that maximizes the

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agent's private value of shirking at t. Finally, the constraint IC_i prevents a deviation in the human capital investment under truth-telling.

By Lemma 1, the social planning problem (P1) can be equivalently expressed as

$$\max_A \ u_0(c_0) + E \sum_{t=1}^{T} \beta^t \left\{ u(c_t) + v(l_t) \right\},$$

$$\text{subject to } (RF), \ \{(IC_{0,t}), (IC_{1,t})\}_{t=1}^{T}, \ (IC_i).$$

(P2)

We now define a relaxed social planning problem (P3):

$$\max_A \ u_0(c_0) + E \sum_{t=1}^{T} \beta^t \left\{ u(c_t) + v(l_t) \right\},$$

$$\text{subject to } (RF), \ \{(IC_{1,t})\}_{t=1}^{T}.$$

(P3)

This problem is identical to the social planning problem (P2), except for the fact that the IC constraints $\{(IC_{0,t})\}_{t=1}^{T}$ and (IC_i) are disregarded. Thus, (P3) is a relaxed version of (P2).

The relaxed planning problem (P3) is a finite-dimensional maximization problem in which the objective function is continuous and the constraint set is closed and bounded. Thus, a solution to (P3) exists. Let $A^* = (i^*, c^*, h^*, l^*, y^*, K^*, Y^*)$ be a solution to the relaxed planning problem (P3), and let $\{j_t^*\}_{t=1}^{T}$ be the associated off-equilibrium human capital investment values defined in (6).

Lemma 2 *In a generic subset of economies, there exists a unique solution to the relaxed planning problem (P3).*[11] *At the solution, all constraints $\{IC_{1,t}\}_{t=1}^{T}$ bind, and the off-equilibrium values $\{j_t^*\}_{t=1}^{T}$ satisfy $j_1^* < j_2^* < ... < j_T^* < i^*$. Moreover, the solution to (P3) is feasible in the social planning problem (P2).*

Proof In Appendix. ∎

The above lemma implies that, generically, there exists a unique optimal allocation in our model, and the optimum can be found as the solution to the relaxed planning problem (P3). At the optimum, the only deviation strategies that bind are the T strategies consisting of the joint action of shirking at t and under-investing in human capital at date 0. Note that these are dynamic deviation strategies: if an agent plans at date 0 to shirk at date $t > 0$, he deviates –already at $t = 0$– from the recommended human capital investment i^* and invests $j_t^* < i^*$. Despite the fact that the deviation

---

[11] The generic set of economies, which we precisely define in Appendix, consists of all economies in which the values $\psi_t(0)$ for $t = 1, ..., T$ are not too large. Essentially, this restriction is made to ensure that over-stating the level of one's human capital is not a binding strategy at any optimum or in any equilibrium implementing the optimum. Having solved many examples numerically, we have yet to find an economy that violates this condition, i.e., does not belong to our generic set. Thus, it is perfectly possible that this assumption is totally innocuous, i.e., that all economies defined in Section 2 satisfy it. Still, we formally restrict the focus to this set of economies because the proofs we present for our analytical results require it.

plan calls for shirking only at date t, due to this under-investment, along the history 1^t, the agent deviates from the recommended labor effort level $l_s^*(1^s) = y_s^*(1^s)/\psi_s(i^*)$ at all dates $s < t$. In fact, at dates $s < t$ the agent overworks because he provides effective labor supply $y_s^*(1^s)$ while his skill level is $\psi_s(j_t^*) < \psi_s(i^*)$.

3.2 Properties of the optimum

This subsection provides a characterization of the optimum in terms of intratemporal, intertemporal, and asset return wedges.

Proposition 1 *(Intratemporal Wedges) At the optimal allocation A^* we have*

1. *a positive intratemporal wedge at all dates for all low-skilled agents, i.e., for all $t = 1, ..., T$ and all η^t such that $h_t(\eta^t) = 0$ we have*

$$-v'(l_t^*(\eta^t)) < u'(c_t^*(\eta^t))\psi_t(0)F_2(K_t^*, Y_t^*), \qquad (8)$$

2. *a negative intratemporal wedge at all non-terminal dates for the high-skilled agents, i.e., for all $t = 1, ..., T - 1$ and all η^t such that $h_t(\eta^t) = i^*$ we have*

$$-v'(l_t^*(1^t)) > u'(c_t^*(1^t))\psi_t(i^*)F_2(K_t^*, Y_t^*), \qquad (9)$$

3. *no intratemporal wedge for the high-skilled at the terminal date*

$$-v'(l_T^*(1^T)) = u'(c_T^*(1^T))\psi_T(i^*)F_2(K_T^*, Y_T^*). \qquad (10)$$

Proof In Appendix. ∎

The positive intratemporal wedge at the bottom of the skill distribution is a standard result in the literature on Mirrlees economies. This wedge is consistent with a positive marginal tax on labor income of the low-skilled. The intuition for the optimality of this wedge is as follows. The binding deviation strategies involve shirking, i.e., the (potential) deviators are over-skilled relative to the truth-tellers who provide the same (low) effective labor supply. Thus, the intratemporal trade-offs between consumption and leisure are different for the deviators and the truth-tellers: the deviators, who over-consume leisure, have a stronger preference for consumption. A positive marginal tax on labor income, therefore, hurts the deviators more than it hurts the truth-tellers, and, thus, relaxes the overall incentive constraint, which makes this intratemporal wedge efficient.

The negative intratemporal wedge at the top of the skill distribution at non-terminal dates is a result that is non-standard. This wedge is consistent with a marginal subsidy to the labor income

of the high-skilled. The intuition for the optimality of this wedge is analogous to the previous case. The deviation strategy of shirking at t is optimally combined with an under-investment in human capital at date 0. Along the path 1^t, the (potential) deviators provide high effective labor supply $y_s^*(1^s)$ for $s = 1, ..., t - 1$ but are under-skilled relative to the agents who follow the equilibrium behavior and provide the same amount of effective labor (because $j_t^* < i^*$). Thus, the deviators have a stronger preference for leisure than the truth-tellers. A marginal subsidy to consumption, thus, helps the deviators less than it helps the truth-tellers, which encourages truth-telling (relaxes the IC constraint) and makes this intratemporal wedge efficient.

Note that all of the binding deviation strategies call for shirking in the last period. Therefore, the observation of $y_T = y_T^*(1^T)$ unambiguously signals a truth-teller (who also has been lucky to receive the high sequence of shocks 1^T). Thus, there is no need for an intratemporal wedge at the top of the skill distribution at $t = T$.

Proposition 2 *(Intertemporal Optimality Conditions) At the optimal allocation A^**

$$\frac{1}{u_0'(c_0^*) + \sum_{t=1}^{T} \alpha_t \left[u_0'(c_0^*) - u_0'(c_0^* + i^* - j_t^*) \right]} = E \left[\frac{1}{r_1^* \beta u'(c_1^*)} \right], \tag{11}$$

where $\alpha_t > 0$ is the Lagrange multiplier associated with the constraint $IC_{1,t}$, and

$$\frac{1}{u'(c_t^*)} = E_t \left[\frac{1}{r_{t+1}^* \beta u'(c_{t+1}^*)} \right], \tag{12}$$

for $t = 1, ..., T - 1$, where $r_{t+1}^* = F_1(K_{t+1}^*, Y_{t+1}^*)$.

Proof In Appendix. ∎

The intertemporal optimality condition (12), which characterizes the optimum at dates $t = 1, ..., T$, is standard in dynamic Mirrlees economies (see Golosov, Kocherlakota, and Tsyvinski 2003). This condition, usually referred to as the Rogerson condition, states that the discounted inverse of the marginal utility of consumption is a martingale at the optimum. Our optimality condition (11), however, is nonstandard. Since, by Lemma 2, $\alpha_t \left[u_0'(c_0^*) - u_0'(c_0^* + i^* - j_t^*) \right] > 0$ for all $t = 1, ..., T$, the intertemporal optimality condition (11) implies the Rogerson condition does not hold at the optimum of our model at date 0. In fact, at date 0, the discounted inverse of the marginal utility of consumption must be a strict supermartingale at the optimum of our model.

Corollary 1 *(Intertemporal Wedges) At the optimal allocation A^**

$$u_0'(c_0^*) < r_1^* \beta E \left[u'(c_1^*) \right] - \sum_{t=1}^{T} \alpha_t \left[u_0'(c_0^*) - u_0'(c_0^* + i^* - j_t^*) \right], \tag{13}$$

where $\alpha_t > 0$ is the Lagrange multiplier associated with the constraint $IC_{1,t}$, and, for $t = 1, ..., T-1$,

$$u'(c_t(1^t)) < r^*_{t+1}\beta E_t \left[u'(c^*_{t+1})| \, 1^t\right].$$
(14)

Proof Follows from the conditions of Proposition 2 by applying the Jensen inequality and using the fact that, for all $t = 1, ..., T$, $c^*_t(1^{t-1}, 0) < c^*_t(1^{t-1}, 1)$ (this fact follows from the fact that all IC constraints $\{IC_{1,t}\}^T_{t=1}$ bind, see proof of Lemma 2). ■

The efficiency of the intertemporal wedge (14), which characterizes the optimal allocation in our model at dates $t = 1, ..., T-1$, follows from the complementarity between shirking tomorrow and saving today. The deviation plans associated with the binding IC constraints call for over-consumption of leisure (shirking), and under-consumption of the consumption good [since $c^*_{t+1}(1^t, 0) < c^*_{t+1}(1^t, 1)$]. Agents who plan to shirk at $t + 1$ would like to save at t more than what the truth-tellers would like to save at t, as the shirkers' marginal utility of consumption at $t + 1$ exceeds the truth-tellers' marginal utility at $t + 1$ state-by-state. By suppressing savings (accumulation of physical capital), the intertemporal wedge, thus, hurts the shirkers more than it hurts the truth-tellers. Suppressed savings, therefore, relax the incentive constraints, which makes the positive intertemporal wedge efficient.

The efficiency of the intertemporal wedge is reinforced at $t = 0$ by the fact that, in addition to over-saving, under-investing in human capital is complementary with shirking. Under any of the T binding deviation strategies, which involve shirking at a future date $t = 1, .., T$ in the life-cycle, agents under-invest in human capital and over-consume already in period zero (as $c^*_0 + i^* - j^*_t > c^*_0$). Thus, those who plan to shirk have a lower marginal utility of consumption at $t = 0$ than those who plan to follow the truthful strategy throughout the life-cycle. Suppressing savings at $t = 0$ increases c^*_0, which benefits the truth-tellers more than it benefits the (potential) shirkers (of all T types). This effect of suppressed savings on current consumption (in addition to the effect of suppressed savings on future consumption) relaxes the incentive compatibility constraints $IC_{1,t}$ for all $t = 1, ..., T$, which reinforces the optimality of the positive intratemporal wedge at period 0.

This additional effect is quantified on the right hand side of (13) by the expression

$$\sum_{t=1}^T \alpha_t \left[u'_0(c^*_0) - u'_0(c^*_0 + i^* - j^*_t)\right] > 0,$$

where $u'_0(c^*_0) - u'_0(c^*_0 + i^* - j^*_t) > 0$ represents the magnitude of slack in the IC constraint $IC_{1,t}$ caused by a marginal increase in c_0, and the Lagrange multiplier $\alpha_t > 0$ represents the welfare value of relaxing the constraint $IC_{1,t}$ by one unit.

Define the intertemporal wedge process $\omega_t : \Theta^t \to \mathbb{R}$ as the missing "implicit tax" in the consumption Euler equation. That is, given an allocation of consumption c and an interest rate sequence r, for each history $\eta^t \in \Theta^t$, $t = 0, 1, \ldots T - 1$, let $\omega_t(\eta^t)$ be defined as the number ω that solves

$$E_t \left[\frac{\beta u'(c_{t+1})}{u'(c_t)} (1 - \omega) r_{t+1} \,\middle|\, \eta^t \right] = 1. \tag{15}$$

The Rogerson property (12) of the optimal allocation c^* at dates $t = 1, \ldots, T - 1$ implies that the optimal intertemporal wedge ω_t^* is nonnegative at all dates $t = 1, \ldots, T - 1$ and states η^t. The inequality (14) implies that $E[\omega_t^*] > 0$ for all $t = 1, \ldots, T - 1$. The modified Rogerson property (11) at $t = 0$ and the inequality (13) provide a tighter lower bound on the optimal intratemporal wedge ω_0^*. Not only is ω_0^* strictly greater than zero, but it is strictly greater than $\sum_{t=1}^{T} \alpha_t \left[1 - u_0'(c_0^* + i^* - j_t^*) / u_0'(c_0^*) \right] > 0$.

In addition to the intra- and inter-temporal wedges, the optimal allocation is characterized by a wedge in the returns on the two types of assets present in our environment. Define time-t return on human capital investment as

$$R_t = \pi^t(1^t) F_2(K_t, Y_t) l_t(1^t) \psi_t'(i).$$

This return measures the additional output obtained at date t due to a marginal increase in human capital investment i at date 0, with all other variables held constant. By R_t^* we denote R_t evaluated at the optimum.

Proposition 3 *(Asset return wedge) At the optimum*

$$\sum_{t=1}^{T} \left(\prod_{s=1}^{t} \frac{1}{r_s^*} \right) R_t^* > 1.$$

Proof In Appendix. ∎

The optimality of a human capital premium follows from the difference between the social incentive costs of human and physical capital investment. As mentioned before, both physical and human capital accumulation are complementary with shirking. An increase in either physical or human capital investment tightens the IC constraints, which has a negative effect on welfare (hence the social incentive cost of investment). However, a given hike in human capital investment tightens the incentive constraints by more than does the same increase in physical capital investment.

Why? The reason is that physical capital investment is observable, while human capital investment is not. Fix an allocation and suppose that \$1 is exogenously added to the economy at date 0 with a recommendation to invest it in physical capital. The only effect on the IC constraints is the standard ex post wealth effect: with more wealth in the future, shirking will be more attractive. There is no effect on incentives in period 0 because physical capital investment is perfectly observ-

able. Now suppose that the extra \$1 comes with a recommendation to invest it in human capital. If the rates of return on physical and human capital investment are equal, the ex post wealth effect on incentives is the same for human as it was for physical capital investment because the extra wealth generated is the same in either case. However, human capital investment is unobservable, i.e., agents can privately divert the recommended human capital investment to consumption at date 0. This additional deviation possibility, which was not available to the agents in the case of physical capital investment, puts an additional strain on the incentive constraints, and thus creates an additional social incentive cost of human capital investment. At the optimum, in order to offset this additional cost, the return on human capital has to exceed the return on physical capital investment, and hence, human capital premium is optimal.

4   Implementation in equilibrium with deferred capital taxes

Human capital is an endogenous, private, and stochastic state variable over the life-cycle of an agent. In section 3, we derived the implications that the presence of this state variable has on the structure of the optimal allocation of capital, labor, and consumption. In this section, we derive the implications of human capital for the structure of optimal capital taxes. Our main finding is the necessity of deferred taxation of capital income in tax systems with linear capital taxes.

In the first subsection below, we formally define competitive equilibrium with taxes and a class of tax systems with linear capital taxes, which we focus our attention on. In the second subsection, we demonstrate that implementation of the optimum is impossible with linear capital taxes if capital can only be taxed contemporaneously, i.e., if all taxes on capital held in period t are due in period t. We then explain how this implementation can be achieved when deferred taxation of capital income is introduced. Finally, in the third subsection, we formally prove our implementation existence result and provide a characterization of an optimal tax system.

4.1   Equilibrium with taxes

In contrast to the direct revelation mechanism used in Section 2 to define and characterize the optimal allocation, we consider here a standard competitive market mechanism in which agents freely trade effective labor, capital, and consumption, subject to taxes. We will call this mechanism a market/tax mechanism.

Agent's problem   All agents are ex ante identical with the initial endowment of capital $k_0 = K_0 - \tilde{\mathcal{T}}_0$, where $\tilde{\mathcal{T}}_0$ is an initial lump-sum tax on each agent. They choose their human capital investment i, initial consumption c_0, savings $k_1 - k_0$, and state-contingent sequences of consumption,

effective labor, and capital, $\{c_t, y_t, k_{t+1}\}_{t=1}^T$, so as to maximize lifetime utility

$$u_0(c_0) + E \sum_{t=1}^T \beta^t \left\{ u(c_t) + v \left(\frac{y_t}{\psi_t(h_t)} \right) \right\}$$

subject to the following set of budget constraints:

$$
\begin{aligned}
c_0 + i + k_1 &\leq k_0, \\
c_t + k_{t+1} &\leq w_t y_t + r_t k_t - \tilde{\phi}_t(y^t) - \tilde{T}_t(y^t, k^t)
\end{aligned}
$$

for $t = 1, ..., T$, where $k_{T+1} = 0$, $\{h_t\}_{t=1}^T$ is the individual human capital process defined in (2)–(3), $\{\tilde{T}_t, \tilde{\phi}_t\}_{t=1}^T$ are the sequences of, respectively, capital and labor income taxes due at time t, and $\{r_t, w_t\}_{t=1}^T$ are the sequences of market gross interest rates and wages.

A class of tax systems Following Kocherlakota (2005), we allow for nonlinear taxation of labor income but restrict attention to taxes linear in capital. Labor income taxes and marginal capital tax rates are allowed to depend on the whole history of labor income. We depart from Kocherlakota (2005), however, by allowing deferred taxation of capital income. This means that the function $\tilde{T}_t(y^t, k^t)$ takes the following form

$$\tilde{T}_t(y^t, k^t) = \sum_{s=1}^t \tilde{\tau}_{s,t}(y^t) r_s k_s,$$

where $\tilde{\tau}_{s,t}(y^t)$ is the marginal tax rate at t on capital that an agent with effective labor history y^t held at $s \leq t$. The tax system used in Kocherlakota (2005) imposes the restriction $\tilde{\tau}_{s,t} = 0$ for all $s < t$.

Competitive equilibrium defined Given a tax system $\tilde{T}_0, \{\tilde{T}_t, \tilde{\phi}_t\}_{t=1}^T$, the notion of competitive equilibrium is standard.

Definition 5 *Given a tax system $\tilde{T}_0, \{\tilde{T}_t, \tilde{\phi}_t\}_{t=1}^T$ and a sequence of government revenue $\{G_t\}_{t=0}^T$, competitive equilibrium is an allocation $A^e = (c_0^e, i^e, \{c_t^e, k_t^e, y_t^e, K_t^e, Y_t^e\}_{t=1}^T)$, and prices $\{r_t, w_t\}_{t=1}^T$ such that:*

1. *given taxes $\{\tilde{T}_t, \tilde{\phi}_t\}_{t=1}^T$ and prices $\{r_t, w_t\}_{t=1}^T$, $(c_0^e, i^e, \{c_t^e, k_t^e, y_t^e\}_{t=1}^T)$ solves the agent's problem;*

2. *prices $\{r_t, w_t\}$ are given by*

$$
\begin{aligned}
r_t &= F_1\left(K_t^e, Y_t^e\right) \\
w_t &= F_2\left(K_t^e, Y_t^e\right)
\end{aligned}
$$

at all $t = 1, ..., T$;

3. *consumption, capital, and effective labor markets clear:*

$$c_0^e + i^e + k_1^e + G_0 = K_0,$$

$$\sum_{\eta^t} \pi^t(\eta^t) c_t^e(\eta^t) + K_{t+1}^e + G_t = F(K_t^e, Y_t^e), \quad t = 1, ...T,$$

$$K_{t+1}^e = \sum_{\eta^t} \pi^t(\eta^t) k_{t+1}^e(\eta^t)), \quad t = 0, ...T-1,$$

$$Y_t^e = \sum_{\eta^t} \pi^t(\eta^t) y_t^e(\eta^t)), \quad t = 1, ...T.$$

A tax system $\tilde{\mathcal{T}}_0$, $\{\tilde{\mathcal{T}}_t, \tilde{\phi}_t\}_{t=1}^T$ is optimal if it implements the optimal allocation A^* as an equilibrium.[12] We will denote an optimal tax system by $\tilde{\mathcal{T}}_0^*$, $\{\tilde{\mathcal{T}}_t^*, \tilde{\phi}_t^*\}_{t=1}^T$.

Expressing taxes in a reduced form Due to the nonlinearity of labor income taxes $\tilde{\phi}_t$ in y^t, the tax system $\{\tilde{\mathcal{T}}_t, \tilde{\phi}_t\}_{t=1}^T$ can introduce an arbitrarily severe punishment on agents whose effective labor supply strategies are such that, for some $t = 1, ..., T$, $y^t \notin \{y^{*t}(\eta^t)\}_{\eta^t \in \Theta^t}$, where $y^{*t} = (y_1^*, ..., y_t^*)$. Assuming each of these detectable deviations is punished severely enough to deter agents from using them, we only need to specify taxes for observed labor income histories y^t such that, for all $t = 1, ..., T$, $y^t = y^{*t}(\eta^t)$ for some $\eta^t \in \Theta^t$. For these histories, we introduce the following notation:

$$\tau_{s,t}(\eta^t) = \tilde{\tau}_{s,t}(y^{*t}(\eta^t)),$$

$$\phi_t(\eta^t) = \tilde{\phi}_t(y^{*t}(\eta^t)).$$

It is therefore sufficient to find reduced-form taxes $\phi_t(\eta^t)$ and

$$\mathcal{T}_t(\eta^t) = \sum_{s=1}^t \tau_{s,t}(\eta^t) r_s k_s,$$

for $t = 1, ..., T$ in order to obtain a characterization of an optimal tax system.

4.2 The necessity of deferred taxation

Before we proceed with our main results in the next subsection, we provide an explanation of why our tax system necessarily needs to use deferred taxes on capital income.

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[12] The variables l and h are not formally included as part of A^e. The equilibrium values of these variables are implied by i^e and y^e.

Suppose that capital taxes are restricted to be contemporaneous: current capital income can be taxed today but not in the future. In particular, capital income obtained in period 1, $r_1 k_1$, can only be taxed in period 1. For the implementation of the optimal allocation in a market equilibrium with taxes to exist, it is necessary that agents do not want to use the markets to trade away from the optimum. In particular, it must be true that the first period Euler equation,

$$u_0'(c_0^*) = r_1 \beta E\left[(1 - \tau_1) u'(c_1^*)\right], \tag{16}$$

is satisfied. Otherwise, agents could improve over the optimum by simply adjusting their savings. At the same time, however, for an implementation with contemporaneous taxes to exist, the following $T - 2$ Euler equations (which are associated with shirking)

$$u_0'(c_0^* + i^* - j_t^*) = r_1 \beta E\left[(1 - \tau_1) u'(c_1^*)\right] \tag{17}$$

must hold for $t = 2, 3, ..., T$. This, however, is impossible as the right-hand sides of (16) and (17) are identical while the left-hand sides differ since, by Lemma 2, $j_2^* < j_3^* < ... < i^*$. Thus, the optimal allocation cannot be implemented with contemporaneous taxes.

Why are conditions (17) necessary for implementation? Suppose that the optimum is implemented in a market/tax mechanism. The equilibrium strategy is to make the initial human capital investment $i^e = i^*$, follow the equilibrium capital accumulation plan k^e, and never shirk. The consumption allocation delivered by this strategy is c^*. If (17) does not hold for some $t = 2, ..., T$, however, this strategy is not individually optimal, and thus it cannot be an equilibrium strategy, and the optimal allocation is not implemented. To see this, consider the private deviation strategy dev_t consisting of shirking in period t, investing in human capital the amount $j_t^* < i^*$, and following the equilibrium physical capital accumulation plan k^e. What value does this strategy deliver in the market/tax mechanism? Under both the optimal direct revelation mechanism and the proposed market/tax mechanism, strategy dev_t yields the same level of expected utility simply because under both mechanisms it generates the same consumption and labor effort plans at all dates and states. In the direct revelation mechanism, dev_t is the strategy that supports the binding IC constraint $IC_{1,t}$. Thus, the utility level delivered by dev_t is equal to that delivered by the optimum. In the proposed implementation mechanism, therefore, agents are indifferent between following the equilibrium strategy and deviating to strategy dev_t. However, dev_t does not exploit the additional dimension of deviation that is, relative to the direct revelation mechanism, available to agents in the market mechanism: deviations of capital holdings k from the proposed equilibrium plan k^e. In particular, if the Euler equation (17) does not hold for t, combining the strategy dev_t with a deviation from $k_1 = k_1^e$ increases the value of dev_t in the proposed market/tax mechanism. Thus, augmenting dev_t

with a deviation along the capital accumulation dimension produces a strategy that yields strictly more utility than the equilibrium strategy does, which contradicts the existence of implementation. Thus, conditions (17) are necessary for implementation.

How does deferred taxation of capital income make implementation of the optimum possible? Suppose that, in addition to being subject to taxes at $t = 1$, capital income $r_1 k_1$ is also taxed in period T. At date T, all human capital risk is resolved and all (indirect) reports about individual realizations of this risk are on record. The marginal tax rate $\tau_{1,T}$ applied at date T to first-date capital income $r_1 k_1$ can use this information, i.e., it can depend on the whole history of reports η^T. The Euler equations associated with truth-telling and shirking in periods $2, ..., T$ are now given by, respectively,

$$u_0'(c_0^*) = r_1 \beta E \left[(1 - \tau_{1,1}) u'(c_1^*) \right] - r_1 \beta^T E \left[\tau_{1,T} u'(c_T^*) \right]$$

and

$$u_0'(c_0^* + i^* - j_t^*) = r_1 \beta E \left[(1 - \tau_{1,1}) r_1 u'(c_1^*) \right] - r_1 \beta^T E \left[\tau_{1,T} u'(c_T^*) \mid \hat{\sigma}_t \right]$$

for $t = 2, ..., T$, where $E \left[\cdot \mid \hat{\sigma}_t \right]$ denotes expectation conditional on shirking strategy t. Deferred tax rates $\tau_{1,T}$ are additional free parameters that may be chosen so as to satisfy all of the $T - 1$ Euler equations above. As none of these Euler equations are colinear, for this to be possible, the terms associated with deferred taxes must be non-colinear. Indeed, they are because under different deviation strategies $\hat{\sigma}_t$ agents arrive at terminal histories η^T with different ex ante probabilities (i.e., $E \left[\cdot \mid \hat{\sigma}_t \right] \neq E \left[\cdot \mid \hat{\sigma}_s \right]$ for $t \neq s$).

Taking as an example the case of $T = 2$, under contemporaneous capital income taxes, the Euler equations for truth-telling and shirking in period 2 are given by, respectively,

$$u_0'(c_0^*) = r_1 \beta E \left[(1 - \tau_{1,1}) u'(c_1^*) \right],$$

and

$$u_0'(c_0^* + i^* - j_2^*) = r_1 \beta E \left[(1 - \tau_{1,1}) u'(c_1^*) \right].$$

These conditions, which we have shown above to be necessary for implementation, cannot be jointly satisfied because $j_2^* < i^*$.

With deferred capital taxes, these Euler equations are given by, respectively,

$$
\begin{aligned}
u_0'(c_0^*) &= r_1 \beta E \left[(1 - \tau_{1,1}) u'(c_1^*) \right] \\
&\quad - r_1 \beta^2 \left[\pi_1(0) \tau_{1,2}(0,0) u'(c_2^*(0,0)) \right] \\
&\quad - r_1 \beta^2 \left[\pi_1(1) \pi_2(0) \tau_{1,2}(1,0) u'(c_2^*(1,0)) \right] \\
&\quad - r_1 \beta^2 \left[\pi_1(1) \pi_2(1) \tau_{1,2}(1,1) u'(c_2^*(1,1)) \right],
\end{aligned}
$$

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and

$$u_0'(c_0^* + i^* - j_2^*) = r_1 \beta E\left[(1 - \tau_{1,1})u'(c_1^*)\right]$$
$$-r_1 \beta^2 \left[\pi_1(0)\tau_{1,2}(0,0)u'(c_2^*(0,0))\right]$$
$$-r_1 \beta^2 \left[\pi_1(1)\tau_{1,2}(1,0)u'(c_2^*(1,0))\right].$$

Both of these conditions can now be supported with an appropriate choice of deferred taxes $\tau_{1,2}$. The deferred tax $\tau_{1,2}(0,0)$ enters both Euler equations with the same coefficient, so making this tax non-zero does not help support these conditions. However, the deferred taxes $\tau_{1,2}(1,0)$ and $\tau_{1,2}(1,1)$ enter the two equations with different coefficients. Setting $\tau_{1,2}(1,0) > \tau_{1,2}(1,1)$ will help bring the two conditions closer together. In particular, both Euler equations are supported if $\tau_{1,2}(0,0) = \tau_{1,2}(1,1) = 0$ and

$$\tau_{1,2}(1,0) = \frac{u_0'(c_0^*) - u_0'(c_0^* + i^* - j_2^*)}{r_1 \beta^2 \pi_1(1)\pi_2(1)u'(c_2^*(1,0))} > 0.$$

Note that the deferred tax $\tau_{1,2}(0,0)$ does not help implementation because the observation of the (indirectly reported) path $(0,0)$ does not carry any information about whether an agent who reports $(0,0)$ follows the deviation strategy dev_2 or tells the truth, as under dev_2 agent lies only in history $(1,1)$. However, an agent who follows strategy dev_2 reports the history $(1,0)$ with probability $\pi_1(1)$, which is more than the true probability $\pi_1(1)\pi_2(0)$. Similar to the standard moral hazard model, this high-likelihood-ratio event is penalized with a high marginal tax rate $\tau_{1,2}(1,0) > \tau_{1,2}(1,1) = 0$. As we have demonstrated, the marginal tax rate $\tau_{1,2}$ on income $r_1 k_1$ must depend on information that becomes available only in the second period of the life-cycle. Thus, deferred taxes are necessary.

4.3 General implementation

In this subsection, we present the main results of our paper, which concern the existence of an implementation and the properties of optimal capital taxes.

Theorem 1 *In a generic class of economies, there exists an optimal tax system $\tilde{\mathcal{T}}_0^*$, $\{\mathcal{T}_t^*, \phi_t^*\}_{t=1}^T$ such that, the contemporaneous capital taxes satisfy*

$$\tau_{t,t}^*(1^t) < 0 \quad for\ 1 \leq t \leq T, \tag{18}$$
$$\tau_{t,t}^*(1^{t-1}, 0) > 0 \quad for\ 1 \leq t \leq T, \tag{19}$$
$$\tau_{t,t}^*(1^{t-s}, 0^s) = 0 \quad for\ 1 < s \leq t \leq T,$$

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and the deferred capital taxes satisfy

$$\tau_{1,t}^*(1^{t-1}, 0) \quad > \quad 0 \quad for \ 1 < t \leq T, \tag{20}$$

$$\tau_{1,t}^*(\eta^t) \quad = \quad 0 \quad for \ 1 < t \leq T \ all \ \eta^t \neq (1^{t-1}, 0), \tag{21}$$

$$\tau_{s,t}^*(\eta^t) \quad = \quad 0 \quad for \ 1 < s < t \leq T \ all \ \eta^t.$$

Proof Constructive. We provide explicit formulas for candidate optimal taxes $\tilde{\mathcal{T}}_0^*$, $\{\mathcal{T}_t^*, \phi_t^*\}_{t=1}^T$ and confirm that the optimal allocation is an equilibrium allocation under these taxes. The generic class of economies is, as before, those economies in which $\psi_t(0)$ for $t = 1, ..., T$ are small enough for lying upward to be strictly suboptimal at the optimum. Details in Appendix. ∎

A key feature of the optimal tax system $\tilde{\mathcal{T}}_0^*$, $\{\mathcal{T}_t^*, \phi_t^*\}_{t=1}^T$ is that the after-tax rate of return on savings is a random variable positively correlated with labor income; despite the fact that savings themselves are riskless. The positive correlation between the return on saving and labor income results from the fact that the marginal tax rate on capital is low for agents with high labor income and high for agents with low labor income. The role for this correlation, as pointed out in Kocherlakota (2005) and Albanesi and Sleet (2006), is to discourage savings just enough to implement the optimal intertemporal wedge. The unique feature of our tax system is that the uncertainty about the marginal capital tax rate is not fully resolved at the time when capital income is realized. In particular, the marginal tax rate on first-period capital income $r_1 k_1$ depends, along some histories, on labor income earned in all periods, including the final date T, as $\tau_{1,T}^*(1^{T-1}, 0) > 0 = \tau_{1,T}^*(1^T)$.

In the environment studied in Albanesi and Sleet (2006), deviations that ultimately shape the structure of optimal capital taxes are static (one-period deviations). Future labor income $w_{t+s} y_{t+s}$, $s \geq 1$, does not carry in this environment any information about agents' current marginal rate of substitution $\beta u'(c_t)/u'(c_{t-1})$ (which is a critical piece of information needed to determine if the intertemporal wedge is satisfied). That the same is true in the environment studied in Kocherlakota (2005) follows directly from Assumption 1 of that paper. In the implementations obtained in Albanesi and Sleet (2006) and Kocherlakota (2005), therefore, taxes on capital income in period t can be contemporaneous, i.e., do not need to be conditioned on labor income from periods $t + 1, t + 2, ...$

In our environment, deviations that bind at the optimum are dynamic: shirking in period t is augmented with under-investment in human capital and over-consumption at date 0. Labor income realized in periods $2, 3, ..., T$ does carry information about the marginal rate of substitution $\beta u'(c_1)/u'(c_0)$. For example, conditional on the observed labor income $w_T y_T = w_T y_T^*(1^T)$, the agent's marginal rate of substitution $\beta u'(c_1)/u'(c_0)$ equals $\beta u'(c_1^*)/u'(c_0^*)$ with probability 1, as only under the truthful strategy agents supply at T effective labor $y_T = y_T^*(1^T)$. Conditional on the observation $w_T y_T = w_T y_T^*(1^{T-1}, 0)$, however, the marginal rate of substitution $\beta u'(c_1)/u'(c_0)$ equals

$\beta u'(c_1^*)/u'(c_0^*)$ with probability $\pi_T(0)/(1+\pi_T(0))$ and $\beta u'(c_1^*)/u'(c_0^*+i^*-j_T^*)$ with probability $1/(1+\pi_T(0))$, as the observation of effective labor supply $y_T = y_T^*(1^{T-1},0)$ is consistent with both the equilibrium strategy and the strategy dev_T, under which agents shirk in the last period.[13] An efficient tax system does not disregard this information. The role for deferred capital taxes, therefore, is to make use of this information and implement the intertemporal wedge efficiently.

Note also that, since the low human capital state $h_t = 0$ is absorbing in our environment, no new information about the agent is released by observations of labor income $w_{t+s}y_{t+s} = w_{t+s}y_{t+s}^*(1^{t-1},0,0^s)$ for $s \geq 1$, i.e., in all periods subsequent to the agent's first (indirect) report of the low human capital level $h_t = 0$. In the optimal tax system of our Theorem 1, capital taxes are zero in all such histories. This feature of the optimal tax system is not necessary, however. There exist other implementations in which capital taxes paid in those histories are non-zero. Intuitively, with history-dependent deferred capital taxation, postponing tax collection can always be done without loss of efficiency, as no information is lost by waiting. In particular, as can be seen in our discussion in the previous sub-section, there exits in our environment an optimal tax system in which all capital taxes are postponed until the terminal date T.[14]

The next proposition provides a further characterization of optimal capital income taxes.

Proposition 4 At the optimal tax system $\tilde{\mathcal{T}}_0^*$, $\{\mathcal{T}_t^*, \phi_t^*\}_{t=1}^T$

$$E\left[\tau_{1,1}^* + \sum_{t=2}^T \left(\prod_{s=2}^t \frac{1}{r_s^*}\right)\tau_{1,t}^*\right] = 0, \tag{22}$$

$$E[\tau_{s,t}^* \mid \eta^{t-1}] = 0, \ \text{for } 1 < t \leq s, \text{ all } \eta^{s-1}. \tag{23}$$

Proof In Appendix. ∎

This proposition shows that the present value of expected capital tax payments due from each agent in this economy is zero, i.e., the amount of government capital income tax revenue is zero. This result is not specific to the implementation $\tilde{\mathcal{T}}_0^*$, $\{\mathcal{T}_t^*, \phi_t^*\}_{t=1}^T$. As we mentioned before, there are other linear capital tax implementations in our environment, which postpone tax collections even more than our implementation $\tilde{\mathcal{T}}_0^*$, $\{\mathcal{T}_t^*, \phi_t^*\}_{t=1}^T$. In all these implementations, government revenue will be zero. This can be seen from the fact that our proof of this proposition follows from the on- and off-equilibrium Euler equations and the (modified) Rogerson conditions which characterize the

---

[13]In this example, the distributions over the possible values of the marginal rate of substitution $\beta u'(c_1)/u'(c_0)$ conditional on the two observations of period-T labor income levels represent posterior beliefs about $\beta u'(c_1)/u'(c_0)$ under the prior distributed uniformly over the truthful strategy and the T deviation strategies, dev_t for $t = 1, ..., T$. The fact that these posteriors do not coincide means that future labor income does carry information about the marginal rate of substitution $\beta u'(c_1)/u'(c_0)$. Along the equilibrium path, of course, all agents follow the truthful strategy. Yet, still, the off-equilibrium beliefs determine the equilibrium outcome.

[14]This non-uniqueness is similar to the indeterminacy of government debt path pointed out in Bassetto and Kocherlakota (2004).

optimum. As we have shown before, any linear implementation must obey the on- and off-equilibrium Euler equations, and, of course, must satisfy the Rogerson conditions, which are not specific to the implementation but rather to the allocation implemented. Therefore, any linear implementation will feature zero expected capital taxes in present value.

This result is intuitive when we take into account that lump-sum taxes are available to the government. In our implementation of the optimum, capital taxes are linear in capital, hence, they are distortionary. With non-distortionary lump-sum taxes available, there is no need to use distortionary capital taxes to raise revenue. The role for capital taxation, in our as well as in other dynamic Mirrlees models, is to provide incentives to the agents, rather than to raise revenue.

Expected capital taxes due from an agent are not zero in every period. In our implementation, agents face an expected subsidy on capital income in period 1, and expected positive capital tax payments due in periods $2, ..., T$. More precisely, the expected capital tax payment due in period 1 is given by

$$E\left[\tau^*_{1,1} r_1 k_1\right] = E\left[\tau^*_{1,1}\right] r_1 k_1 < 0,$$

where the strict inequality follows from (22) and (20). The expected capital tax payment due in period $t = 2, .., T$ is given by

$$
\begin{aligned}
E\left[\tau^*_{1,t} r_1 k_1 + \tau^*_{t,t} r_t k_t\right] &= E\left[\tau^*_{1,t}\right] r_1 k_1 + E\left[\tau^*_{t,t}\right] r_t k_t \\
&= E\left[\tau^*_{1,t}\right] r_1 k_1 > 0,
\end{aligned}
$$

where the second equality follows from (23) and the strict inequality follows from (20).

4.4 Marginal tax rate volatility

In this subsection, we demonstrate how the large intertemporal wedge that characterizes the optimal allocation in our economy early in the life cycle (i.e., at date 0) translates into large volatility of marginal capital tax rates in the implementation. As our benchmark, we take an exogenous-skill version of our environment. We show that, relative to the exogenous-skill environment, the intertemporal wedge in our environment is large. Then, we show how this translates into larger volatility of marginal tax rates needed to implement the optimum in our endogenous-skill model, relative to the volatility needed for implementation in the benchmark exogenous-skill model.

Suppose that the skill process ψ_t is exogenously fixed and there is no human capital investment in the initial period of the life-cycle. This environment is a special case of the environment studied in Golosov, Kocherlakota, Tsyvinski (2003) and Kocherlakota (2005). Our incentive compatibility

constrains $IC_{1,t}$, given in (5), reduce in this case to

$$u\left(c_t(1^t)\right) + v(l_t(1^t)) \geq u\left(c_t(1^{t-1},0)\right) + v\left(\frac{\psi_t(1^{t-1},0)l_t(1^{t-1},0)}{\psi_t(1^t)}\right)$$

for $t = 1, .., T$. The optimal allocation of consumption, denoted $\hat{c}$, satisfies the standard Rogerson condition

$$\frac{1}{u'(\hat{c}_t)} = E_t\left[\frac{1}{r_{t+1}\beta u'(\hat{c}_{t+1})}\right] \tag{24}$$

at all dates, including $t = 0$.

In contrast to the endogenous-skill environment in which we have $T+1$ Euler equations at $t = 1$, the exogenous-skill model has only two Euler conditions at $t = 1$ that need to be satisfied in an implementation with linear capital taxes: the on-equilibrium Euler condition

$$u'_0(\hat{c}_0) = r_1\beta E\left[(1 - \tau_1)u'(\hat{c}_1)\right],$$

and the off-equilibrium condition

$$u'_0(\hat{c}_0) = r_1\beta(1 - \tau_1(0))u'(\hat{c}_1(0)).$$

Solving the off-equilibrium Euler equation for $\tau_1(0)$ and the on-equilibrium equation for $\tau_1(1)$, we obtain optimal marginal capital tax rates, denoted $\hat{\tau}_1$, given by

$$\hat{\tau}_1(\theta) = 1 - \frac{u'_0(\hat{c}_0)}{\hat{r}_1\beta u'(\hat{c}_1(\theta))} \tag{25}$$

for $\theta \in \Theta$, where $\hat{r}_1$ denotes the optimal gross interest rate in the economy with exogenous skills.

For the purpose of the comparison with the exogenous-skill model, define the total marginal capital tax rate at $t = 1$ in the endogenous-skill model to be the present value of contemporaneous and deferred marginal tax rates on capital income r_1k_1. The intertemporal wedge ω is defined in (15) .

Proposition 5 *Consider an endogenous-skill economy and an exogenous-skill Mirrlees economy with the same preferences over consumption. Suppose that the same consumption allocation is optimal in both economies, i.e., $\hat{c} = c^*$. Then, the intratemporal wedge at $t = 0$ and the volatility of the marginal capital tax rate at $t = 1$ are strictly larger in the economy with endogenous skills.*

Proof In Appendix. ∎

This proposition tell us that, keeping the volatility of the consumption process constant, the volatility of the marginal tax rate depends on the underlying friction. In the proof, we show that

the shadow interest rate at $t = 1$ is larger in the endogenous-skill economy, i.e., $r_1^* > \hat{r}_1$. Given that the consumption allocations are the same in the two compared economies, this immediately implies that $\omega_1^* > \hat{\omega}_1$. This larger wedge translates into larger volatility of total marginal tax rate on first period capital income needed to implement the same consumption allocation $c^* = \hat{c}$ under the more severe friction of the endogenous skill economy.

5   A numerical example

In this section, we use a parameterized example to explore numerically aspects of the optimal allocation and the tax system studied in pervious sections. We take a model period of 10 years. Agents begin life at age 15, and work from the age of 25 until 65. The period between the ages of 15 and 25 is when agents have the human capital investment opportunity but do not work. We assume that the distribution of agents' human capital investment shock is given by $\pi_0(1) = \pi_0(0) = .5$ and the conditional distributions of human capital depreciation shocks σ_t are $\pi_t(1) = \pi_t(0) = .5$ for all $t \geq 1$. The skill function ψ_t is given by

$$\psi_t(h_t) = a_t + b_t \sqrt{h_t}$$

for $t = 1, ..., 5$ with constants

$$
\begin{aligned}
(a_1, ..., a_5) &= (0.2, 0.35, 0.4, 0.3, 0.25), \\
(b_1, ..., b_5) &= (1, 1.5, 2.3, 2, 1.5).
\end{aligned}
$$

The utility functions are taken to be $u_0 = u = \log$, and $v(l) = -l^2$. We set a discount factor β to 0.8, which implies an annual discount factor of 0.98. The aggregate production function is given by $F(K_t, Y_t) = rK_t + wY_t$, $K_0 = 1$, where we set $r = 1/\beta$, and $w = 1$.

Figure 1 presents the low skill profile (at $h_t = 0$) and the high skill profile at the optimal human capital investment (i.e., at $h_t = i^*$). Figure 2 displays the optimal intratemporal wedges across the realized histories η^t. We observe that the intratemporal wedge at the top of the skill distribution is negative and the absolute size of this wedge decreases with the age of the high-skilled agent. Figure 3 displays the intertemporal wedge at the initial period and in periods $t = 2, ..., 5$ conditional on undepreciated human capital in period $t - 1$, i.e., along the history $\eta^6 = 1^6$ (for all the other realized types, this wedge is equal to zero). As we observe, in this example, the intertemporal wedge declines with the agent's age. Intuitively, the incentive problem is most severe early in the life-cycle when the private human capital investment is made, which translates into a large intertemporal wedge in this period. Finally, Figure 4 shows the optimal contemporaneous capital taxes for the high skilled as

well as the optimal contemporaneous and deferred capital taxes for the agents whose human capital just depreciated. As we observe, the optimal marginal deferred tax rate decreases with the duration of human capital.

6 Conclusion

Deferred taxes are a common feature of capital income tax systems currently used in many countries. Our paper provides a theoretical rationale for the use of such solutions. Our results show that it is necessary to use deferred taxes in settings in which information relevant for the assessment of tax is revealed gradually over time.

We show that when human capital accumulation is taken into account in a way consistent with three main empirical facts about the life-cycle properties of individual-specific human capital, the problem of optimal taxation of individual income constitutes an important example of a setting in which deferred taxes must be used. In a Mirrlees economy in which human capital investment is private, risky, and non-separable from consumption, a three-way complementarity between shirking, under-investing in human capital, and over-saving requires that a portion of tax on capital income obtained by agents early in the life-cycle be deferred until late in the life-cycle, when more information about agents' private human capital decisions is available through the observation of longer labor income histories. Long histories of high labor income are consistent with high effort and high human capital investment. The deferred tax assessed on agents with such observed histories is low. Histories of low labor income, in contrast, are consistent with over-consumption and under-investment in human capital early in the life-cycle and shirking at later dates in the life cycle. Therefore, the deferred tax assessed on agents with such observed histories is high.

Our results do not depend on several assumptions that we make for the ease of exposition. First, we assume that all agents are ex ante identical in our model. Our results go through with minor changes when ex ante agent heterogeneity is incorporated into the model, as long as these differences in individual characteristics are publicly observable. Second, the model can be easily modified to replace the period-by-period resource constraint with the present value resource constraint. Third, in the market implementation we consider, capital is the only asset that agents trade. If bond markets with observable trades are introduced into the model, our results go through without change, with all wealth (physical capital and financial claims) receiving the same tax treatment.

Appendix

Before we proceed with the proof of Lemma 1, we prove an auxiliary lemma. Consider the agent's choice of investment and reporting strategy given in the IC constraint of Definition 3. Lemma A1 below shows that if in this maximization problem a one-period overstating of the level of human capital is not a profitable deviation from truth-telling at any t, then no lying strategy that consists of multiple-period overstatements of human capital can be profitable.

Lemma A1   *For each $t = 1, ..., T$, conditions $\{IC_{0,s}\}_{s=t}^{T}$ imply*

$$w_t(i, (1^{t-1}, 0), (1^{t-1}, 0)) \geq w_t(i, (1^{t-1}, 0), (1^{t-1}, 1)).$$

Proof of Lemma A1   Directly from the definition of w_T (given in Definition 3), $IC_{0,T}$ is the same condition as

$$w_T(i, (1^{T-1}, 0), (1^{T-1}, 0)) \geq w_T(i, (1^{T-1}, 0), (1^{T-1}, 1)). \tag{26}$$

Thus, we have our conclusion for $t = T$.

At $T - 1$ we have

$$
\begin{aligned}
w_{T-1}(i, (1^{T-2}, 0), (1^{T-2}, 0)) \quad = \quad & u\left(c_{T-1}(1^{T-2}, 0)\right) + v\left(l_{T-1}(1^{T-2}, 0)\right) \\
& + \beta \left\{ u\left(c_T(1^{T-2}, 0^2)\right) + v\left(l_T(1^{T-2}, 0^2)\right) \right\},
\end{aligned}
\tag{27}
$$

and

$$
\begin{aligned}
w_{T-1}(i, (1^{T-2}, 0), (1^{T-2}, 1)) \quad = \quad & u\left(c_{T-1}(1^{T-2}, 1)\right) + v\left(\frac{\psi_{T-1}(i) l_{T-1}(1^{T-2}, 1)}{\psi_{T-1}(0)}\right) \\
& + \beta \max_{\hat{\sigma}_{T-1}} \left\{ w_T(i, (1^{T-2}, 0^2), (1^{T-2}, 1, \hat{\sigma}_{T-1})) \right\}.
\end{aligned}
\tag{28}
$$

Note now that

$$w_T(i, (1^{T-2}, 0^2), (1^{T-2}, 1, \hat{\sigma}_{T-1})) = w_T(i, (1^{T-1}, 0), (1^{T-1}, \hat{\sigma}_{T-1})) \tag{29}$$

as both sides of this equation are equal to

$$u(c_T(1^{T-1}, \hat{\sigma}_{T-1})) + v\left(\frac{\psi_T(i\hat{\sigma}_{T-1}) l_T(1^{T-1}, \hat{\sigma}_{T-1})}{\psi_T(0)}\right).$$

Now, (29) and (26) imply that

$$\max_{\hat{\sigma}_{T-1}} \left\{ w_T(i, (1^{T-2}, 0^2), (1^{T-2}, 1, \hat{\sigma}_{T-1})) \right\} = w_T(i, (1^{T-2}, 0^2), (1^{T-2}, 1, 0))$$
$$= u(c_T(1^{T-1}, 0)) + v\left(l_T(1^{T-1}, 0)\right).$$

We can thus rewrite (28) as

$$w_{T-1}(i, (1^{T-2}, 0), (1^{T-2}, 1)) = u\left(c_{T-1}(1^{T-2}, 1)\right) + v\left(\frac{\psi_{T-1}(i)l_{T-1}(1^{T-2}, 1)}{\psi_{T-1}(0)}\right)$$
$$+ \beta u(c_T(1^{T-1}, 0)) + v\left(l_T(1^{T-1}, 0)\right),$$

Substituting this equality and (27) to the desired inequality

$$w_{T-1}(i, (1^{T-2}, 0), (1^{T-2}, 0)) \geq w_{T-1}(i, (1^{T-2}, 0), (1^{T-2}, 1)),$$

we obtain $IC_{0,T-1}$, which yields our conclusion for $t = T - 1$.

Replicating the same argument for $t = T - 2, T - 3, ..., 2, 1$, we get the desired conclusion for all $t = 1, ..., T$. $\square$

Proof of Lemma 1

Necessity If allocation A is IC, then the condition $IC_{0,t}$ must hold for all $t = 1, ..., T$. Suppose it does not for some t. Consider the following investment-announcement strategy for the agent: invest in human capital the recommended amount i, truthfully announce all shocks to human capital up to time $t - 1$ and then, if $h_{t-1} = i$ and $\sigma_{t-1} = 0$ (i.e., if $h_t = 0$ for the first time), declare $\hat{\sigma}_{t-1} = 1$ and, in the following period, $\hat{\sigma}_t = 0$. This strategy of one-period over-statement of skill would yield more utility to the agent than the truthful investment-revelation strategy, which violates the assumed incentive compatibility of the allocation A.

If allocation A is IC, then the condition $IC_{1,t}$ must hold for all $t = 1, ..., T$. Suppose it does not for some t. Consider the following investment-announcement strategy for the agent: invest in human capital the amount j_t given in (6), consume the difference $i - j_t > 0$ at $t = 0$, truthfully announce all shocks to human capital up to time $t - 1$ and then, if $h_t = j_t$ (i.e., if human capital remains non-zero in period t, although lower than the on-equilibrium amount i), declare $\hat{\sigma}_{t-1} = 0$. This strategy of under-investing in human capital and over-consuming in period zero, followed by a false report of zero human capital in period t yields more utility to the agent than the truthful investment-revelation strategy, which violates the assumed incentive compatibility of the allocation A.

If allocation A is IC, then the condition IC_i must hold. Suppose it does not. Consider the following investment-announcement strategy for the agent: invest in human capital the amount $j \neq i$ which does solve IC_i, adjust consumption at $t = 0$ by $i - j$, and truthfully announce all shocks to human capital at all periods $t = 1, ..., T$. This strategy of mis-investing in human capital period zero, with no false reporting of human capital shocks, yields more utility to the agent than the truthful investment-revelation strategy, which violates the assumed incentive compatibility of the allocation A.

Sufficiency We show that if allocation A is not IC, then at least one of the conditions IC_i, $IC_{0,t}$ or $IC_{1,t}$ for some $t = 1, ..., T$ must be violated at A. Let $(j, \hat{\eta}^T)$ be an investment-reporting strategy which, under the allocation A, delivers more utility to the agent than the strategy (i, η^T) of investing the recommended amount and reporting the shocks truthfully. Call any such strategy an upsetting strategy. Allocation A is not IC if an upsetting strategy exists. We show that if there exists an upsetting strategy, then at least one of the conditions IC_i, $IC_{0,t}$ or $IC_{1,t}$ for some $t = 1, ..., T$ must be violated at A.

Fix an upsetting strategy $(j, \hat{\eta}^T)$. First, suppose that $\hat{\eta}^T = \eta^T$, i.e., suppose that $(j, \hat{\eta}^T)$ involves only mis-investment and no lying about the realized history of shocks. With no lying, the upsetting strategy and the equilibrium strategy (i, η^T) imply the same effective labor and consumption assignments at all histories η^t and all dates $t = 1, ..., T$. The difference in utility value of these two strategies comes from a) the over-consumption at date 0 by the amount $i - j$, and b) a disutility of labor difference along the path 1^T, at which the amount actually invested in human capital matters. Thus, since (j, η^T) is upsetting, we have

$$u_0(c_0 + i - j) + \sum_{t=1}^{T} \beta^t \pi^t(1^t) v\left(\frac{\psi_t(i) l_t(1^t)}{\psi_t(j)}\right) > u_0(c_0) + \sum_{t=1}^{T} \beta^t \pi^t(1^t) v\left(l_t(1^t)\right).$$

Denote the left-hand side of the above inequality by $V(j)$, which represents the value of private human capital investment j under truth-telling. The above inequality says that the recommended investment level i does not maximize V. Note also that, as u, v, ψ_t are all strictly concave, V is a strictly concave function of j. The condition IC_i is a first-order (FO) condition $V'(j) = 0$ evaluated at i. Since, i does not maximize V, this FO condition must be violated.

Suppose then that $\hat{\eta}^T \neq \eta^T$, i.e., that there is an upsetting strategy $(j, \hat{\eta}^T)$ that involves lying about the realized history of shocks. Let t be the time when the state gets misreported for the first time under $(j, \hat{\eta}^T)$. Thus, $\hat{\eta}^{t-1} = \eta^{t-1}$. Also, it must be the case that $\eta^{t-1} = 1^{t-1}$. If $\eta^{t-1} \neq 1^{t-1}$, then $1(\hat{\eta}^{t-1}) = 1(\eta^{t-1}) = 0$ and, thus, the set of reports available at t is $\Sigma_t(\hat{\eta}^{t-1}) = \{0\}$, which makes lying for the first time in period t impossible. Thus, there are two possible histories of length t for which the first lie can occur: $(1^{t-1}, 1)$ and $(1^{t-1}, 0)$, each associated with one possible

misrepresentation: $(1^{t-1}, 0)$ and $(1^{t-1}, 1)$, respectively.

Consider first the case of reporting $\hat{\eta}^t = (1^{t-1}, 0)$ when $\eta^t = (1^{t-1}, 1)$ and $\hat{\eta}^t = (1^{t-1}, 0)$ when $\eta^t = (1^{t-1}, 0)$, i.e., the case in which the agent "lies down" by under-reporting the realized shock if $\sigma_{t-1} = 1$ but tells the truth if $\sigma_{t-1} = 0$. The report $\hat{\sigma}_{t-1} = 0$ determines all subsequent reports as $\hat{\sigma}_s = 0$ for $s \geq t$. Thus, the complete reporting strategy associated with this misrepresentation is to tell the truth in periods $s = 1, ...t-1$, and in period t announce $\hat{\sigma}_{t-1} = 0$ for both $\sigma_{t-1} \in \Theta$, given that $\Sigma_t(\eta^{t-1}) = \{0, 1\}$. The inequality $IC_{1,t}$ requires that the equilibrium strategy yields at least as much utility as this reporting strategy does under the initial investment level that maximizes the value of this reporting strategy, i.e., j_t. Thus, $IC_{1,t}$ must be violated because this reporting strategy, together with some level of human capital investment j, is upsetting.

Consider now the case of the first lie at t after history $\eta^t = (1^{t-1}, 0)$ with no lying after history $(1^{t-1}, 1)$. There are $T - t$ complete reporting strategies associated with this misrepresentation. Since we assumed no lying before t or after history $(1^{t-1}, 1)$, and state $\sigma_{t-1} = 0$ is absorbing, human capital of an agent whose $\sigma_{t-1} = 0$ is zero at all remaining dates $t + 1, t + 2, ..., T$. However, given the lie $\hat{\sigma}_{t-1} = 1$, there are $T - t$ remaining dates in the life-cycle at each of which the agent can either keep up the lie by continuing to report high shock realizations or reveal the low shock (i.e., low human capital). The $T - t$ complete reporting strategies that feature the first lie at $\eta^t = (1^{t-1}, 0)$ and no lying at or after $(1^{t-1}, 1)$ are then as follows: if $\eta^{t-1} = 1^{t-1}$ and $\sigma_{t-1} = 0$, then report the high shock for s periods, and admit the low skill after s consecutive skill overstatements, where $s = 1, 2, ..., T - t$; otherwise report truthfully. We now show that if any of these reporting plans, combined with some initial level of human capital investment j, constitutes an upsetting strategy, then at least one of the inequalities $IC_{1,s}$ for $s = t, ..., T$ or IC_i must be violated. There are two possibilities: either $w_t(j, (1^{t-1}, 0), (1^{t-1}, 1)) > w_t(j, (1^{t-1}, 0), (1^{t-1}, 0))$ or not. If not, then replacing the sequence of lies after the history $(1^{t-1}, 0)$ with truthtelling results in an investment-reporting plan that also is upsetting. But this plan involves truthtelling throughout, so IC_i must be violated, as shown above. Consider then the case in which

$$w_t(j, (1^{t-1}, 0), (1^{t-1}, 1)) > w_t(j, (1^{t-1}, 0), (1^{t-1}, 0)). \tag{30}$$

The shock $\sigma_{t-1} = 0$ erases all human capital investment, so the continuation value $w_t(j, (1^{t-1}, 0), (1^{t-1}, \hat{\sigma}_{t-1}))$ does not depend on j. Thus (30) implies

$$w_t(i, (1^{t-1}, 0), (1^{t-1}, 1)) > w_t(i, (1^{t-1}, 0), (1^{t-1}, 0)).$$

By Lemma A1, this strict inequality implies that one of the inequalities $IC_{1,s}$ for $s = t, ..., T$ must be violated.

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The last set of strategies that we need to consider consists of those that involve both reporting $\hat{\eta}^t = (1^{t-1}, 0)$ when $\eta^t = (1^{t-1}, 1)$ and $\hat{\eta}^t = (1^{t-1}, 1)$ when $\eta^t = (1^{t-1}, 0)$ with some lying horizon $s \leq T - t$. If a strategy of this form is upsetting, then either the strategy of lying only in history $(1^{t-1}, 1)$ or in history $(1^{t-1}, 0)$ must be upsetting, too. We have shown already that both cases lead to a violation of the IC conditions.

Thus, we have that if there exists (among all possible investment-reporting strategies) an upsetting strategy, then at least one of the IC conditions IC_i or $IC_{x,t}$ for some $x \in \Theta$, $t = 1, ..., T$ must be violated. Thus, these conditions are sufficient for overall incentive compatibility of an allocation A. $\square$

Proof of Lemma 2

First, we show the following lemma.

Lemma A2 *At any solution $A^* = (i^*, c^*, h^*, l^*, y^*, K^*, Y^*)$ to problem P3*

$$c_s^*(1^t, \eta^{s-t}) \geq c_s^*(1^{t-1}, 0, 0^{s-t}) \tag{31}$$

for all $t = 1, ..., T$ and all $s \geq t$ and $\eta^{s-t} \in \Theta^{s-t}$.

Proof of Lemma A2 Suppose to the contrary that

$$c_{\hat{s}}^*(1^{\hat{t}}, \eta^{\hat{s}-t}) < c_{\hat{s}}^*(1^{\hat{t}-1}, 0, \eta^{\hat{s}-t}) \tag{32}$$

a some $\hat{t} \leq \hat{s}$ and $\eta^{\hat{s}-\hat{t}} \in \Theta^{\hat{s}-\hat{t}}$. Consider the allocation $\bar{A} = (i^*, \bar{c}, h^*, l^*, y^*, K^*, Y^*)$, where $\bar{c} = c^*$ for $\eta^t \notin \{(1^{\hat{t}}, \eta^{\hat{s}-t}), (1^{\hat{t}-1}, 0, \eta^{\hat{s}-t})\}$, and where

$$\bar{c}_{\hat{s}}(1^{\hat{t}}, \hat{\eta}^{\hat{s}-\hat{t}}) = c_{\hat{s}}^*(1^{\hat{t}}, \hat{\eta}^{\hat{s}-\hat{t}}) + \varepsilon \tag{33}$$

$$\bar{c}_{\hat{s}}(1^{\hat{t}-1}, 0, 0^{\hat{s}-\hat{t}}) = c_{\hat{s}}^*(1^{\hat{t}-1}, 0, 0^{\hat{s}-\hat{t}}) - \frac{\pi_{\hat{s}}(1^{\hat{t}}, \hat{\eta}^{\hat{s}-\hat{t}})}{\pi_{\hat{s}}(1^{\hat{t}-1}, 0, 0^{\hat{s}-\hat{t}})} \varepsilon \tag{34}$$

for a small $\varepsilon > 0$. Clearly, $\bar{A}$ is resource feasible. Also, by the Envelope Theorem, the sequence of off-equilibrium investment levels $\{j_t\}_{t=1}^T$ associated with $\bar{A}$ coincides with the values $\{j_t\}_{t=1}^T$ associated with A^*. Let I_t^* denote the slack in the IC constraint $IC_{1,t}$ at allocation A^* and $\bar{I}_t$ denote the slack in the IC constraint $IC_{1,t}$ at the allocation $\bar{A}$. By feasibility of A^* in P3, $I_t^* \geq 0$ for all t. We now show that $\bar{I}_t \geq I_t^* (\geq 0)$ for all t, which means that $\bar{A}$ is feasible in P3.

Because of the way consumption levels $c_{\hat{s}}(1^{\hat{t}}, \hat{\eta}^{\hat{s}-\hat{t}})$ and $c_{\hat{s}}(1^{\hat{t}-1}, 0, 0^{\hat{s}-\hat{t}})$ enter the IC constraints, we consider two cases.

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Case 1: $\hat{s} = \hat{t}$. The ad absurdum assumption (32) reduces to

$$c_{\hat{t}}^*(1^{\hat{t}}) < c_{\hat{t}}^*(1^{\hat{t}-1}, 0) \tag{35}$$

and (33) and (34) reduce to

$$
\begin{aligned}
\bar{c}_{\hat{t}}(1^{\hat{t}}) &= c_{\hat{t}}^*(1^{\hat{t}}) + \varepsilon \\
\bar{c}_{\hat{t}}(1^{\hat{t}-1}, 0) &= c_{\hat{t}}^*(1^{\hat{t}-1}, 0) - \frac{\pi_{\hat{t}}(1)}{\pi_{\hat{t}}(0)}\varepsilon.
\end{aligned}
$$

Since $\bar{c}_{\hat{t}}(1^{\hat{t}})$ and $\bar{c}_{\hat{t}}(1^{\hat{t}-1}, 0)$ do not show up in the constraints $IC_{1,t}$ for $t > \hat{t}$, we have that $\bar{I}_t = I_t^* \geq 0$ for $t > \hat{t}$.

For $t = \hat{t}$, since $\bar{c}_{\hat{t}}(1^{\hat{t}})$ enters the IC constraint $IC_{1,\hat{t}}$ on the left-hand side (LHS) and $\bar{c}_{\hat{t}}(1^{\hat{t}-1}, 0)$ enters on the RHS, transferring a small amount to those who declare $1^{\hat{t}}$ clearly relaxes the IC constraint $IC_{1,\hat{t}}$. More formally, we have

$$
\begin{aligned}
\bar{I}_t &= I_t^* + \beta^{\hat{t}}\pi^{\hat{t}}(1^{\hat{t}})\left\{\left[u(\bar{c}_{\hat{t}}(1^{\hat{t}})) - u(c_{\hat{t}}^*(1^{\hat{t}}))\right] - \left[u(\bar{c}_{\hat{t}}(1^{\hat{t}-1}, 0)) - u(c_{\hat{t}}^*(1^{\hat{t}-1}, 0))\right]\right\} \\
&= I_t^* + \beta^{\hat{t}}\pi^{\hat{t}}(1^{\hat{t}})\left\{\left[u(c_{\hat{t}}^*(1^{\hat{t}}) + \varepsilon) - u(c_{\hat{t}}^*(1^{\hat{t}}))\right] - \left[u\left(c_{\hat{t}}^*(1^{\hat{t}-1}, 0) - \frac{\pi_{\hat{t}}(1)}{\pi_{\hat{t}}(0)}\varepsilon\right) - u(c_{\hat{t}}^*(1^{\hat{t}-1}, 0))\right]\right\} \\
&= I_t^* + \beta^{\hat{t}}\pi^{\hat{t}}(1^{\hat{t}})\left\{\left[u(c_{\hat{t}}^*(1^{\hat{t}}) + \varepsilon) - u(c_{\hat{t}}^*(1^{\hat{t}}))\right] + \left[u(c_{\hat{t}}^*(1^{\hat{t}-1}, 0)) - u\left(c_{\hat{t}}^*(1^{\hat{t}-1}, 0) - \frac{\pi_{\hat{t}}(1)}{\pi_{\hat{t}}(0)}\varepsilon\right)\right]\right\} \\
&> I_t^*.
\end{aligned}
$$

where the strict inequality follows from the fact that u is increasing.

For $t < \hat{t}$, $\bar{c}_{\hat{t}}(1^{\hat{t}})$ and $\bar{c}_{\hat{t}}(1^{\hat{t}-1}, 0)$ show up only on the LHS of the IC constraints $IC_{1,t}$. Therefore (using Taylor approximation),

$$
\begin{aligned}
\bar{I}_t &= I_t^* + \beta^{\hat{t}}\left\{\pi^{\hat{t}}(1^{\hat{t}})\left[u(\bar{c}_{\hat{t}}(1^{\hat{t}})) - u(c_{\hat{t}}^*(1^{\hat{t}}))\right] + \pi^{\hat{t}}(1^{\hat{t}-1}, 0)\left[u(\bar{c}_{\hat{t}}(1^{\hat{t}-1}, 0)) - u(c_{\hat{t}}^*(1^{\hat{t}-1}, 0))\right]\right\} \\
&= I_t^* + \beta^{\hat{t}}\left\{\pi^{\hat{t}}(1^{\hat{t}})\left[u(c_{\hat{t}}^*(1^{\hat{t}}) + \varepsilon) - u(c_{\hat{t}}^*(1^{\hat{t}}))\right] + \pi^{\hat{t}}(1^{\hat{t}-1}, 0)\left[u\left(c_{\hat{t}}^*(1^{\hat{t}-1}, 0) - \frac{\pi_{\hat{t}}(1)}{\pi_{\hat{t}}(0)}\varepsilon\right) - u(c_{\hat{t}}^*(1^{\hat{t}-1}, 0))\right]\right\} \\
&= I_t^* + \beta^{\hat{t}}\left\{\pi^{\hat{t}}(1^{\hat{t}})\left[u'(c_{\hat{t}}^*(1^{\hat{t}}))\varepsilon\right] - \pi^{\hat{t}}(1^{\hat{t}-1}, 0)\left[u'\left(c_{\hat{t}}^*(1^{\hat{t}-1}, 0)\right)\frac{\pi_{\hat{t}}(1)}{\pi_{\hat{t}}(0)}\varepsilon\right]\right\} \\
&= I_t^* + \beta^{\hat{t}}\pi^{\hat{t}}(1^{\hat{t}})\varepsilon\left\{u'(c_{\hat{t}}^*(1^{\hat{t}})) - u'\left(c_{\hat{t}}^*(1^{\hat{t}-1}, 0)\right)\right\} \\
&> I_t^*
\end{aligned}
$$

where the strict inequality follows from (35). This strict inequality also implies that welfare attained by $\bar{A}$ is strictly greater than that attained by A^*, which contradicts the assumption that A^* solves P3.

Case 2: $\hat{s} > \hat{t}$. Note that $c_{\hat{s}}(1^{\hat{t}}, \hat{\eta}^{\hat{s}-\hat{t}})$ and $c_{\hat{s}}(1^{\hat{t}-1}, 0, 0^{\hat{s}-\hat{t}})$ enter the IC constraints $IC_{1,t}$ only

for $t < \hat{t}$. For any date $t < \hat{t}$ and a history $\hat{\eta}^{\hat{s}-\hat{t}} \in \Theta^{\hat{s}-\hat{t}}$, consumption levels $c_{\hat{s}}(1^{\hat{t}}, \hat{\eta}^{\hat{s}-\hat{t}})$ and $c_{\hat{s}}(1^{\hat{t}-1}, 0, 0^{\hat{s}-\hat{t}})$ show up in the constraint $IC_{1,t}$ only once each, both with the same coefficient, $\beta^{\hat{s}} \sum_{\eta^{\hat{s}-\hat{t}}} \pi^{\hat{s}}(1^{\hat{t}}, \hat{\eta}^{\hat{s}-\hat{t}})$, with $c_{\hat{s}}(1^{\hat{t}}, \hat{\eta}^{\hat{s}-\hat{t}})$ entering on the LHS of $IC_{1,t}$ and $c_{\hat{s}}(1^{\hat{t}-1}, 0, 0^{\hat{s}-\hat{t}})$ entering on the RHS of $IC_{1,t}$. Thus, transferring a small amount to those who declare $(1^{\hat{t}}, \hat{\eta}^{\hat{s}-\hat{t}})$ ambiguously relaxes the IC constraint $IC_{1,t}$. We, therefore, get $\bar{I}_t \geq I_t^*$ for all t. The argument showing that welfare attained by $\bar{A}$ is strictly greater than that attained by A^* is identical to the one presented in case 1 above. Thus, we get the desired contradiction in Case 2, as well. $\square$

Lemma A2 implies that it is without loss of generality to disregard allocations that violate the weak spread condition (31). More precisely, any solution to P3 also solves a maximization problem P3' which is constructed by imposing (31) as an additional constraint in P3.

We now restrict attention to a generic subset $\mathcal{E}_0$ of the set of all economies we have defined in Section 2. We define $\mathcal{E}_0$ as the set of all economies in which the values $\psi_t(0)$ are sufficiently close to zero for $t = 1, ..., T$ so that it is true that at any solution to problem P3'

$$y_t^*(1^t) > y_t^*(1^{t-s}, 0^s),$$

and $y_t^*(1^{t-s}, 0^s)$ is close to zero for all $t = 1, ..., T$ and $s \leq t$.

We now show the following lemma.

Lemma A3 *For all economies in $\mathcal{E}_0$, the constraint set of problem P3' is convex.*

Proof of Lemma A2 Let I_t^n denote the slack in the IC constraint $IC_{1,t}$ at allocation $A^n = (i^n, c^n, h^n, l^n, y^n, K^n, Y^n)$ for $n \in \{1, 2, \alpha\}$, where A^1 and A^2 are two allocations feasible in P3' and A^α is a linear combination of A^1 and A^2 with $\alpha \in [0, 1]$ being the weight on A^1. Clearly, A^α is resource feasible. By the feasibility of A^1 and A^2 in P3', $I_t^n \geq 0$ for $n = 1, 2$ all $t = 1, .., T$. We need to show that $I_t^\alpha \geq 0$ for all $t = 1, .., T$.

In order to do so, we first derive a first-order Taylor approximation of I_t at any allocation feasible in P3'. Bringing all terms in the condition $IC_{1,t}$, given in (5), to the LHS, we get

$$
\begin{aligned}
I_t &= [u_0(c_0) - u_0(c_0 + i - j_t)] + \sum_{s=1}^{t-1} \beta^s \pi^s(1^s) \left[v(l_s(1^s)) - v\left(\frac{\psi_s(i) l_s(1^s)}{\psi_s(j_t)} \right) \right] \\
&\quad + \beta^t \pi^t(1^t) \left[u\left(c_t(1^t)\right) - u\left(c_t(1^{t-1}, 0)\right) \right] + \beta^t \pi^t(1^t) \left[v(l_t(1^t)) - v\left(\frac{\psi_t(0) l_t(1^{t-1}, 0)}{\psi_t(j_t)} \right) \right] \\
&\quad + \sum_{s=t+1}^{T} \beta^s \sum_{\eta^{s-t}} \pi^s(1^t, \eta^{s-t}) \left[u(c_s(1^t, \eta^{s-t})) - u(c_s(1^{t-1}, 0, 0^{s-t})) \right] \\
&\quad + \sum_{s=t+1}^{T} \beta^s \sum_{\eta^{s-t}} \pi^s(1^t, \eta^{s-t}) \left[v(l_s(1^t, \eta^{s-t})) - v\left(\frac{\psi_s(0) l_s(1^{t-1}, 0, 0^{s-t})}{\psi_s(j_t 1(\eta^{s-t}))} \right) \right]
\end{aligned}
$$

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where $\{j_t\}_{t=1}^T$ satisfy (6). Using effective labor supply $y_t = \psi_t l_t$ we can rewrite I_t equivalently as follows

$$
\begin{aligned}
I_t \;=\;& [u_0(c_0) - u_0(c_0 + i - j_t)] + \sum_{s=1}^{t-1} \beta^s \pi^s(1^s)\left[v\left(\frac{y_s(1^s)}{\psi_s(i)}\right) - v\left(\frac{y_s(1^s)}{\psi_s(j_t)}\right)\right] \\
&+ \beta^t \pi^t(1^t)\left[u\left(c_t(1^t)\right) - u\left(c_t(1^{t-1},0)\right)\right] + \beta^t \pi^t(1^t)\left[v\left(\frac{y_t(1^t)}{\psi_t(i)}\right) - v\left(\frac{y_t(1^{t-1},0)}{\psi_t(j_t)}\right)\right] \\
&+ \sum_{s=t+1}^T \beta^s \sum_{\eta^{s-t}} \pi^s(1^t,\eta^{s-t})\left[u(c_s(1^t,\eta^{s-t})) - u(c_s(1^{t-1},0,0^{s-t}))\right] \\
&+ \sum_{s=t+1}^T \beta^s \sum_{\eta^{s-t}} \pi^s(1^t,\eta^{s-t})\left[v\left(\frac{y_s(1^t,\eta^{s-t})}{\psi_s(i)}\right) - v\left(\frac{y_s(1^{t-1},0,0^{s-t})}{\psi_s(j_t 1(\eta^{s-t}))}\right)\right].
\end{aligned}
$$

Regrouping terms, we get

$$
\begin{aligned}
I_t \;=\;& [u_0(c_0) - u_0(c_0 + i - j_t)] + \sum_{s=1}^{t-1} \beta^s \pi^s(1^s)\left[v\left(\frac{y_s(1^s)}{\psi_s(i)}\right) - v\left(\frac{y_s(1^s)}{\psi_s(j_t)}\right)\right] \\
&+ \beta^t \pi^t(1^t)\left[v\left(\frac{y_t(1^t)}{\psi_t(i)}\right) - v\left(\frac{y_t(1^{t-1},0)}{\psi_t(j_t)}\right)\right] \\
&+ \sum_{s=t+1}^T \beta^s \pi^s(1^s)\left[v\left(\frac{y_s(1^t,\eta^{s-t})}{\psi_s(i)}\right) - v\left(\frac{y_s(1^{t-1},0,0^{s-t})}{\psi_s(j_t)}\right)\right] \\
&+ \sum_{s=t+1}^T \beta^s \sum_{\eta^{s-t} \neq 1^{s-t}} \pi^s(1^t,\eta^{s-t})\left[v\left(\frac{y_s(1^t,\eta^{s-t})}{\psi_s(i)}\right) - v\left(\frac{y_s(1^{t-1},0,0^{s-t})}{\psi_s(0)}\right)\right] \\
&+ \beta^t \pi^t(1^t)\left[u\left(c_t(1^t)\right) - u\left(c_t(1^{t-1},0)\right)\right] \\
&+ \sum_{s=t+1}^T \beta^s \sum_{\eta^{s-t}} \pi^s(1^t,\eta^{s-t})\left[u(c_s(1^t,\eta^{s-t})) - u(c_s(1^{t-1},0,0^{s-t}))\right].
\end{aligned}
$$

Replacing the differences in square brackets with their Taylor approximations around the points in

the second term of each bracket, we get

$$
\begin{aligned}
I_t =& \left[u_0'(c_0 + i - j_t)(i - j_t)\right] + \sum_{s=1}^{t-1} \beta^s \pi^s(1^s) \left[\frac{d}{dj_t} v\left(\frac{y_s(1^s)}{\psi_s(j_t)}\right)(i - j_t)\right] \\
&+ \beta^t \pi^t(1^t) \left[\frac{\partial}{\partial j_t} v\left(\frac{y_t(1^{t-1}, 0)}{\psi_t(j_t)}\right)(i - j_t)\right] + \beta^t \pi^t(1^t) \left[\frac{\partial}{\partial y_t} v\left(\frac{y_t(1^{t-1}, 0)}{\psi_t(j_t)}\right)(y_t(1^t) - y_t(1^{t-1}, 0))\right] \\
&+ \sum_{s=t+1}^{T} \beta^s \pi^s(1^s) \left[\frac{\partial}{\partial j_t} v\left(\frac{y_s(1^{t-1}, 0, 0^{s-t})}{\psi_s(j_t)}\right)(i - j_t)\right] \\
&+ \sum_{s=t+1}^{T} \beta^s \pi^s(1^s) \left[\frac{\partial}{\partial y_s} v\left(\frac{y_s(1^{t-1}, 0, 0^{s-t})}{\psi_s(j_t)}\right)(y_s(1^t, \eta^{s-t})) - y_s(1^{t-1}, 0, 0^{s-t}))\right] \\
&+ \sum_{s=t+1}^{T} \beta^s \sum_{\eta^{s-t} \neq 1^{s-t}} \pi^s(1^t, \eta^{s-t}) \left[\frac{\partial}{\partial i} v\left(\frac{y_s(1^{t-1}, 0, 0^{s-t})}{\psi_s(i)}\right)\bigg|_{i=0} i\right] \\
&+ \sum_{s=t+1}^{T} \beta^s \sum_{\eta^{s-t} \neq 1^{s-t}} \pi^s(1^t, \eta^{s-t}) \left[\frac{\partial}{\partial y_s} v\left(\frac{y_s(1^{t-1}, 0, 0^{s-t})}{\psi_s(0)}\right)(y_s(1^t, \eta^{s-t})) - y_s(1^{t-1}, 0, 0^{s-t}))\right] \\
&+ \beta^t \pi^t(1^t) u'\left(c_t(1^{t-1}, 0)\right)(c_t(1^t) - c_t(1^{t-1}, 0)) \\
&+ \sum_{s=t+1}^{T} \beta^s \sum_{\eta^{s-t}} \pi^s(1^t, \eta^{s-t}) u'(c_s(1^{t-1}, 0, 0^{s-t}))(c_s(1^t, \eta^{s-t}) - c_s(1^{t-1}, 0, 0^{s-t})).
\end{aligned}
$$

Adding up the terms that involve $i - j_t$ and factoring $i - j_t$ out, we get that the expression multiplying $i - j_t$ is identical to the LHS of (6), thus equal zero. The terms that are left are

$$
\begin{aligned}
I_t =& \beta^t \pi^t(1^t) \left[\frac{\partial}{\partial y_t} v\left(\frac{y_t(1^{t-1}, 0)}{\psi_t(j_t)}\right)(y_t(1^t) - y_t(1^{t-1}, 0))\right] \\
&+ \sum_{s=t+1}^{T} \beta^s \pi^s(1^s) \left[\frac{\partial}{\partial y_s} v\left(\frac{y_s(1^{t-1}, 0, 0^{s-t})}{\psi_s(j_t)}\right)(y_s(1^t, \eta^{s-t})) - y_s(1^{t-1}, 0, 0^{s-t}))\right] \\
&+ \sum_{s=t+1}^{T} \beta^s \sum_{\eta^{s-t} \neq 1^{s-t}} \pi^s(1^t, \eta^{s-t}) \left[\frac{\partial}{\partial i} v\left(\frac{y_s(1^{t-1}, 0, 0^{s-t})}{\psi_s(i)}\right)\bigg|_{i=0} i\right] \\
&+ \sum_{s=t+1}^{T} \beta^s \sum_{\eta^{s-t} \neq 1^{s-t}} \pi^s(1^t, \eta^{s-t}) \left[\frac{\partial}{\partial y_s} v\left(\frac{y_s(1^{t-1}, 0, 0^{s-t})}{\psi_s(0)}\right)(y_s(1^t, \eta^{s-t})) - y_s(1^{t-1}, 0, 0^{s-t}))\right] \\
&+ \beta^t \pi^t(1^t) u'\left(c_t(1^{t-1}, 0)\right)(c_t(1^t) - c_t(1^{t-1}, 0)) \\
&+ \sum_{s=t+1}^{T} \beta^s \sum_{\eta^{s-t}} \pi^s(1^t, \eta^{s-t}) u'(c_s(1^{t-1}, 0, 0^{s-t}))(c_s(1^t, \eta^{s-t}) - c_s(1^{t-1}, 0, 0^{s-t})).
\end{aligned}
$$

In the generic set of economies $\mathcal{E}_0$, we have $y_t(1^{t-1}, 0)$, $y_s(1^{t-1}, 0, 0^{s-t})$, $y_s(1^{t-1}, 0, 0^{s-t})$ close to zero, so, under some regularity conditions on the boundary behavior of derivatives $v'(0)$ and $\psi_t'(0)$,

the above expression can be approximated by

$$
\begin{aligned}
I_t \;=\;& \beta^t \pi^t(1^t)\left[v'(0)\,y_t(1^t)\right] \\
&+ \sum_{s=t+1}^{T} \beta^s \pi^s(1^s)\left[v'(0)\,y_s(1^t,\eta^{s-t})\right] \\
&+ \sum_{s=t+1}^{T} \beta^s \sum_{\eta^{s-t}\neq 1^{s-t}} \pi^s(1^t,\eta^{s-t})\left[v'(0)\,i\right] \\
&+ \sum_{s=t+1}^{T} \beta^s \sum_{\eta^{s-t}\neq 1^{s-t}} \pi^s(1^t,\eta^{s-t})\left[v'(0)\,y_s(1^t,\eta^{s-t}))\right] \\
&+ \beta^t \pi^t(1^t) u'\left(c_t(1^{t-1},0)\right)\left(c_t(1^t)-c_t(1^{t-1},0)\right) \\
&+ \sum_{s=t+1}^{T} \beta^s \sum_{\eta^{s-t}} \pi^s(1^t,\eta^{s-t}) u'(c_s(1^{t-1},0,0^{s-t}))(c_s(1^t,\eta^{s-t})-c_s(1^{t-1},0,0^{s-t})).
\end{aligned}
$$

Given that the terms in the first four lines are linear, in order to show that $I_t^\alpha \geq 0$ when $I_t^n \geq 0$ for $n=1,2$, it is sufficient to show that

$$
\begin{aligned}
&u'\left(c_s^\alpha(1^{t-1},0,0^{s-t})\right)\left(c_s^\alpha(1^t,\eta^{s-t})-c_s^\alpha(1^{t-1},0,0^{s-t})\right) \\
\geq\; & \left[au'\left(c_s^1(1^{t-1},0,0^{s-t})\right)+(1-\alpha)u'\left(c_s^2(1^{t-1},0,0^{s-t})\right)\right]\left(c_s^\alpha(1^t,\eta^{s-t})-c_s^\alpha(1^{t-1},0,0^{s-t})\right)(36)
\end{aligned}
$$

for all $t=1,...,T$ and all $s \geq t$ and $\eta^{s-t} \in \Theta^{s-t}$.

By Lemma A2, $c_s^n(1^t,\eta^{s-t})-c_s^n(1^{t-1},0,0^{s-t}) \geq 0$ for $n=1,2$, all $t=1,...,T$ all $s \geq t$ and $\eta^{s-t} \in \Theta^{s-t}$. Thus,

$$
c_s^\alpha(1^t,\eta^{s-t})-c_s^\alpha(1^{t-1},0,0^{s-t}) \geq 0
$$

for all $t=1,...,T$ all $s \geq t$ and $\eta^{s-t} \in \Theta^{s-t}$. Thus, dividing through by $c_s^n(1^t,\eta^{s-t})-c_s^n(1^{t-1},0,0^{s-t})$, we get that (36) holds iff

$$
u'\left(c_s^\alpha(1^{t-1},0,0^{s-t})\right) \geq \left[au'\left(c_s^1(1^{t-1},0,0^{s-t})\right)+(1-\alpha)u'\left(c_s^2(1^{t-1},0,0^{s-t})\right)\right],
$$

i.e., iff u' is concave, which is true by the NIARA of u. $\square$

By Lemma A3, the constraint set in P3' is convex. Thus, P3' is a strictly concave maximization problem, i.e., it has a unique maximum, which satisfies the first-order conditions (FOC) of P3'. By Lemma 2, this maximum also is the maximum in problem P3.[15]

We now proceed to proving the conclusions of Lemma 2.

---

[15] If at any solution to P3', $y_t^*(\eta^t)$ is weakly monotone in the number of ones in η^t, then an analog of Lemma A3 can be shown to hold true also for economies outside $\mathcal{E}_0$ under the additional assumption of concavity of v', i.e., $v''' < 0$. Restricting attention to the generic set $\mathcal{E}_0$ makes the proofs less tedious. Numerically, we have not found an example of an economy for which any of the conclusions we draw in the generic case would not hold.

We first show that the values $\{j_t^*\}_{t=1}^T$ satisfy $j_1^* < j_2^* < ... < j_T^*$.

From definition (6), given that, generically, $y_t^*(1^t) > 0$ and $y_t^*(1^{t-s}, 0^s) \simeq 0$ for all $t = 1, ..., T$ and $s \leq t$, j_t^* is the unique maximizer of the function $f_t : \mathbb{R} \to \mathbb{R}$ defined as

$$f_t(j) = u_0(c_0^* + i^* - j) + \sum_{s=1}^{t-1} \beta^s \pi^s(1^s) v\left(\frac{y_s^*(1^s)}{\psi_s(j)}\right) \tag{37}$$

for $t = 1, ..., T$. (The uniqueness of j_t^* follows from the fact that f_t is strictly concave.) Given that $u_0' > 0$, $v' < 0$, and $\psi_s' > 0$ for $s = 1, ..., T$, it is immediate from (37) that $j_t^* < j_{t+1}^*$ for all $t = 1, ..., T$.

Note also that, since $f_t > f_{t+1}$ for $t = 1, ..., T - 1$, we have that

$$f_t(j_t^*) > f_{t+1}(j_{t+1}^*) \tag{38}$$

for $t = 1, ..., T - 1$.

We now show that at the solution to the relaxed planning problem P3, all IC constraints $\{IC_{1,t}\}_{t=1}^T$ bind.

Suppose that $IC_{1,T}$ is slack at the solution $A^* = (i^*, c^*, h^*, l^*, y^*, K^*, Y^*)$. The Lagrange multiplier associated with this constraint, α_T, equals zero. The first-order (FO) necessary conditions with respect to $c_t(1^t)$ and $c_t(1^{t-1}, 0)$ are given, respectively, by

$$\beta^t u'(c_t(1^t)) \left[1 + \sum_{s=1}^t \alpha_s\right] = \lambda_t \tag{39}$$

and

$$\beta^t u'(c_t(1^{t-1}, 0)) \left[1 + \sum_{s=1}^{t-1} \alpha_s - \alpha_t \frac{\pi^t(1^{t-1}, 1)}{\pi^t(1^{t-1}, 0)}\right] = \lambda_t \tag{40}$$

for $t = 1, ..., T$, where $\sum_{s=1}^0$ denotes the empty sum. These FO conditions for $t = T$, imply that $c_T^*(1^T) = c_T^*(1^{T-1}, 0)$ when $\alpha_T = 0$. Also, given that $\psi_T(0)$ is close to zero, generically, we have $l_T^*(1^T) > l_T^*(1^{T-1}, 0) \simeq 0$. Consider now the investment-reporting strategy of investing i^* and reporting the truth except in history 1^T, in which $(1^{T-1}, 0)$ is reported (shirking in period T). Given that

$$u(c_T^*(1^T)) + v(l_T^*(1^T)) < u(c_T^*(1^{T-1}, 0)) + v(l_T^*(1^{T-1}, 0)),$$

this strategy upsets the truthful investment-revelation strategy. Thus, the strategy of investing j_T^* and shirking in period T upsets the truthful strategy even more, as j_T^* is the level of i that maximizes the value of this reporting strategy under the allocation A^*. But this means that the constraint $IC_{1,T}$ is violated, which contradicts the supposition that it is slack. Thus, $IC_{1,T}$ binds.

Note also that, in the generic case, the binding IC condition $IC_{1,T}$ can be written out as

$$u_0(c_0^*) + \sum_{s=1}^{T-1} \beta^s \pi^s(1^s) v(l_s^*(1^s)) + \beta^T \pi^T(1^T) \left\{ u\left(c_T^*(1^T)\right) + v(l_T^*(1^T)) \right\}$$

$$= u_0(c_0^* + i^* - j_T^*) + \sum_{s=1}^{T-1} \beta^s \pi^s(1^s) v\left(\frac{\psi_s(i^*)l_s^*(1^s)}{\psi_s(j_T^*)} \right) + \beta^T \pi^T(1^T) \left\{ u\left(c_T^*(1^{T-1},0)\right) + 0 \right\}$$

or, equivalently, as

$$u_0(c_0^*) + \sum_{s=1}^{T-1} \beta^s \pi^s(1^s) v(l_s^*(1^s)) + \beta^T \pi^T(1^T) \left\{ u\left(c_T^*(1^T)\right) + v(l_T^*(1^T)) - u\left(c_T^*(1^{T-1},0)\right) \right\}$$

$$= u_0(c_0^* + i^* - j_T^*) + \sum_{s=1}^{T-1} \beta^s \pi^s(1^s) v\left(\frac{\psi_s(i^*)l_s^*(1^s)}{\psi_s(j_T^*)} \right)$$

$$= f_T(j_T^*), \tag{41}$$

where the last equality uses definition (37).

Suppose now that $IC_{1,T-1}$ is slack, with $\alpha_{T-1} = 0$. The FO conditions (39) and (40) for $t = T-1$ imply that $c_{T-1}^*(1^{T-1}) = c_{T-1}^*(1^{T-2},0)$. The slack condition $IC_{1,T-1}$ can then be written as

$$u_0(c_0^*) + \sum_{s=1}^{T-2} \beta^s \pi^s(1^s) v(l_s^*(1^s)) + \beta^{T-1} \pi^{T-1}(1^{T-1}) \left\{ u\left(c_{T-1}^*(1^{T-1})\right) + v(l_t^*(1^t)) \right\}$$

$$+ \beta^T \pi^{T-1}(1^{T-1}) \sum_{\sigma_T} \pi_T(\sigma_T) \left\{ u(c_T^*(1^{T-1},\sigma_T)) + v(l_T^*(1^{T-1},\sigma_T)) \right\} >$$

$$u_0(c_0^* + i^* - j_{T-1}^*) + \sum_{s=1}^{T-2} \beta^s \pi^s(1^s) v\left(\frac{\psi_s(i^*)l_s^*(1^s)}{\psi_s(j_{T-1}^*)} \right)$$

$$+ \beta^{T-1} \pi^{T-1}(1^{T-1}) \left\{ u\left(c_{T-1}^*(1^{T-2},0)\right) + v(l_{T-1}^*(1^{T-2},0)) \right\}$$

$$+ \beta^T \pi^{T-1}(1^{T-1}) \sum_{\sigma_T} \pi_T(\sigma_T) \left\{ u(c_T^*(1^{T-1},0)) + v(l_T^*(1^{T-1},0)) \right\}.$$

Given that $c_{T-1}^*(1^{T-1}) = c_{T-1}^*(1^{T-2},0)$ under our supposition and using the fact that, in the generic case, the terms after the history $(1^{T-1},0)$ cancel out, we can rewrite this inequality as

$$u_0(c_0^*) + \sum_{s=1}^{T-2} \beta^s \pi^s(1^s) v(l_s^*(1^s)) + \beta^{T-1} \pi^{T-1}(1^{T-1}) v(l_t^*(1^t))$$

$$+ \beta^T \pi^T(1^T) \left\{ u(c_T^*(1^T)) + v(l_T^*(1^T)) - u(c_T^*(1^{T-1},0)) \right\}$$

$$> u_0(c_0^* + i^* - j_{T-1}^*) + \sum_{s=1}^{T-2} \beta^s \pi^s(1^s) v\left(\frac{\psi_s(i^*)l_s^*(1^s)}{\psi_s(j_{T-1}^*)} \right)$$

$$= f_{T-1}(j_{T-1}^*). \tag{42}$$

Using (41), the above strict inequality implies that $f_{T-1}(j_{T-1}^*) < f_T(j_T^*)$ which contradicts (38). Thus $IC_{1,T-1}$ binds.

Repeating the same argument for $t = T-2, ..., 1$, we get that at the solution to the relaxed planner problem all the IC constraints $\{IC_{1,T}\}_{t=1}^T$ bind.

We now proceed to showing that the solution to (P3) is feasible in (P2).

That $\{IC_{0,t}\}_{t=1}^T$ are satisfied at the solution to (P3) obvious in the generic class of economies in which $\psi_t(0)$ is close to zero for $t = 1, ..., T$ (as the left-hand sides of $IC_{0,t}$ involve the term $y_t^*(1^{t-1}, 1)/\psi_t(0)$, which is very large when $\psi_t(0)$ is small).

In order to show that the solution to (P3) satisfies IC_i we use the FO conditions of the relaxed planning problem (P3), which characterize the solution. In particular, the first order (FO) conditions with respect to c_0, i, and $l_t(1^t)$ for $t = 1, ..., T$ are as follows:

$$u_0'(c_0) + \sum_{t=1}^T \alpha_t \left(u_0'(c_0) - u_0'(c_0 + i - j_t) \right) = \lambda_0 \tag{43}$$

$$-\sum_{t=1}^T \alpha_t u_0'(c_0 + i - j_t) = \sum_{t=2}^T \alpha_t \left[\sum_{s=1}^{t-1} \beta^s \pi^s(1^s) v' \left(\frac{\psi_s(i) l_s(1^s)}{\psi_s(j_t)} \right) \frac{\psi_s'(i) l_s(1^s)}{\psi_s(j_t)} \right]$$
$$+ \lambda_0 - \sum_{t=1}^T \lambda_t \pi^t(1^t) \psi_t'(i) l_t(1^t) F_2(K_t, Y_t) \tag{44}$$

$$\pi^t(1^t) \beta^t v'(l_t) \frac{1}{\psi_t(i)} \left[1 + \sum_{s=1}^T \alpha_s \right] - \pi^t(1^t) \beta^t \sum_{s=t+1}^T \alpha_s v' \left(\frac{\psi_t(i) l_t(1^t)}{\psi_t(j_s)} \right) \frac{1}{\psi_t(j_s)}$$
$$= -\lambda_t \pi^t(1^t) F_2(K_t, Y_t), \tag{45}$$

where, as before, $\alpha_t \geq 0$ is the Lagrangian multiplier of $IC_{1,t}$ and $\lambda_t \geq 0$ is the multiplier associated with the time-t resource constraint. In the second term of (45) for $t = T$, as well as elsewhere in the paper, $\sum_{T+1}^T$ denotes the empty sum (a sum of zero components).

Combining equations (43) and (44) we get

$$\left[1 + \sum_{t=1}^T \alpha_t \right] u_0'(c_0)$$
$$= \sum_{t=1}^T \lambda_t \pi^t(1^t) \psi_t'(i) l_t(1^t) F_2(K_t, Y_t) - \sum_{t=2}^T \alpha_t \left[\sum_{s=1}^{t-1} \beta^s \pi^s(1^s) v' \left(\frac{\psi_s(i) l_s(1^s)}{\psi_s(j_t)} \right) \frac{\psi_s'(i) l_s(1^s)}{\psi_s(j_t)} \right].$$

Substituting in this equation $\lambda_t \pi^t(1^t) F_2(K_t, Y_t)$ from the conditions (45) yields

$$\left[1 + \sum_{t=1}^{T} \alpha_t \right] u_0'(c_0) =$$

$$-\sum_{t=2}^{T} \alpha_t \left[\sum_{s=1}^{t-1} \beta^s \pi^s(1^s) v' \left(\frac{\psi_s(i) l_s(1^s)}{\psi_s(j_t)} \right) \frac{\psi_s'(i) l_s(1^s)}{\psi_s(j_t)} \right] - \left[1 + \sum_{t=1}^{T} \alpha_t \right] \sum_{t=1}^{T} \pi^t(1^t) \beta^t v'(l_t) \frac{\psi_t'(i) l_t(1^t)}{\psi_t(i)}$$

$$+ \sum_{t=1}^{T-1} \pi^t(1^t) \beta^t \sum_{s=t+1}^{T} \alpha_s v' \left(\frac{\psi_t(i) l_t(1^t)}{\psi_t(j_s)} \right) \frac{\psi_t'(i) l_t(1^t)}{\psi_t(j_s)} \qquad (46)$$

Using the fact that

$$\sum_{t=2}^{T} \alpha_t \left[\sum_{s=1}^{t-1} \beta^s \pi^s(1^s) v' \left(\frac{\psi_s(i) l_s(1^s)}{\psi_s(j_t)} \right) \frac{\psi_s'(i) l_s(1^s)}{\psi_s(j_t)} \right] = \sum_{t=1}^{T-1} \pi^t(1^t) \beta^t \sum_{s=t+1}^{T} \alpha_s v' \left(\frac{\psi_t(i) l_t(1^t)}{\psi_t(j_s)} \right) \frac{\psi_t'(i) l_t(1^t)}{\psi_t(j_s)},$$

we cancel out terms in (46) and get

$$u_0'(c_0) = -\sum_{t=1}^{T} \beta^t \pi^t(1^t) v' \left(l_t(1^t) \right) \frac{\psi_t'(i) l_t(1^t)}{\psi_t(i)}.$$

This necessary condition on the solution to (P3) coincides with the constraint IC_i. Thus, we conclude that the solution to the relaxed planning problem (P3) satisfies the IC constraint IC_i.

Finally, $i^* > j_T^*$ follows from the fact that i^* satisfies (7), j_T^* satisfies (6) and $\pi^T(1^T) y_T^*(1^T) > 0$. $\square$

Proof of Proposition 1

The FO conditions with respect to $c_s(1^{t-1}, 0, 0^{s-t})$, $l_t(1^t)$, $l_t(1^{t-1}, 0)$, and $l_s(1^{t-1}, 0, 0^{s-t})$ for all t, s such that $1 \leq t < s \leq T$ are, respectively, as follows:

$$\beta^s u'(c_s(1^{t-1}, 0, 0^{s-t})) \left[1 + \sum_{n=1}^{t-1} \alpha_n - \alpha_t \sum_{\eta^{s-t}} \frac{\pi^s(1^{t-1}, 1, \eta^{s-t})}{\pi^s(1^{t-1}, 0, 0^{s-t})} \right] = \lambda_s, \qquad (47)$$

$$\beta^t v'(l_t(1^t)) \left[1 + \sum_{s=1}^{T} \alpha_s \right] - \beta^t \sum_{s=t+1}^{T} \alpha_s v' \left(\frac{\psi_t(i) l_t(1^t)}{\psi_t(j_s)} \right) \frac{\psi_t(i)}{\psi_t(j_s)} =$$

$$-\psi_t(i) \lambda_t F_2(K_t, Y_t), \qquad (48)$$

$$\beta^t v'(l_t(1^{t-1}, 0)) \left[1 + \sum_{s=1}^{t-1} \alpha_s \right] - \alpha_t \frac{\pi^t(1^{t-1}, 1)}{\pi^t(1^{t-1}, 0)} \beta^t v' \left(\frac{\psi_t(0) l_t(1^{t-1}, 0)}{\psi_t(j_t)} \right) \frac{\psi_t(0)}{\psi_t(j_t)} =$$

$$-\psi_t(0) \lambda_t F_2(K_t, Y_t), \tag{49}$$

$$\beta^s v'(l_s(1^{t-1}, 0, 0^{s-t})) \left[1 + \sum_{n=1}^{t-1} \alpha_n \right]$$

$$-\alpha_t \beta^s \sum_{\eta^{s-t}} \frac{\pi^s(1^{t-1}, 1, \eta^{s-t})}{\pi^s(1^{t-1}, 0, 0^{s-t})} v' \left(\frac{\psi_s(0) l_s(1^{t-1}, 0, 0^{s-t})}{\psi_s(j_t 1(\eta^{s-t}))} \right) \frac{\psi_s(0)}{\psi_s(j_t 1(\eta^{s-t}))} =$$

$$\psi_s(0) \lambda_s F_2(K_s, Y_s), \tag{50}$$

where, again, $\sum_{T+1}^{T}$ and $\sum_{1}^{0}$ denote the empty sum.

Note now that, since $v' < 0$, $v'' < 0$, and $\psi_t' > 0$ for all t, we have that, for all t and $y_t > 0$

$$v' \left(\frac{y_t}{\psi_t(j)} \right) \frac{1}{\psi_t(j)} \tag{51}$$

is a strictly increasing function of j.

The FO (48) for $t = 1, ..., T - 1$ evaluated at the optimum can be written as follows

$$\beta^t \sum_{s=t+1}^{T} \alpha_s \left[v' \left(\frac{\psi_t(i^*) l_t^*(1^t)}{\psi_t(j_s^*)} \right) \frac{\psi_t(i^*)}{\psi_t(j_s^*)} - v' \left(\frac{\psi_t(i^*) l_t^*(1^t)}{\psi_t(i^*)} \right) \right] =$$

$$\beta^t v'(l_t^*(1^t)) \left[1 + \sum_{s=1}^{t} \alpha_s \right] + \psi_t(i^*) \lambda_t F_2(K_t^*, Y_t^*). \tag{52}$$

Given that $j_s^* < i^*$ and $\alpha_s > 0$ for all s, and the fact that (51) is strictly increasing, we have that

$$v' \left(\frac{\psi_t(i^*) l_t^*(1^t)}{\psi_t(j_s^*)} \right) \frac{\psi_t(i^*)}{\psi_t(j_s^*)} - v' \left(\frac{\psi_t(i^*) l_t^*(1^t)}{\psi_t(i^*)} \right) < 0$$

for all $s, t = 1, ..., T$ and thus

$$\beta^t \sum_{s=t+1}^{T} \alpha_s \left[v' \left(\frac{\psi_t(i^*) l_t^*(1^t)}{\psi_t(j_s^*)} \right) \frac{\psi_t(i^*)}{\psi_t(j_s^*)} - v' \left(\frac{\psi_t(i^*) l_t^*(1^t)}{\psi_t(i^*)} \right) \right] < 0$$

for all $t = 1, ..., T - 1$. Thus, we get from (52) that

$$-\beta^t v'(l_t^*(1^t)) \left[1 + \sum_{s=1}^{t} \alpha_s \right] > \psi_t(i^*) \lambda_t F_2(K_t^*, Y_t^*)$$

for all $t = 1, ..., T - 1$. Using (39) evaluated at the optimum, we cancel out $\lambda_t \beta^{-t} [1 + \sum_{s=1}^{t} \alpha_s]^{-1}$

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to obtain

$$-v'(l_t^*(1^t)) > \psi_t(i^*)F_2(K_t^*, Y_t^*)u'(c_t^*(1^t))$$

for all $t = 1, ..., T - 1$, which concludes the proof of (9).

The FO (48) for $t = T$ evaluated at the optimum reads simply

$$\beta^T v'(l_T^*(1^T)) \left[1 + \sum_{s=1}^T \alpha_s\right] = -\psi_T(i^*)\lambda_T F_2(K_T^*, Y_T^*).$$

Using (39) for $t = T$ evaluated at the optimum, we cancel out the term $\lambda_T \beta^{-T}[1 + \sum_{s=1}^T \alpha_s]^{-1}$ to obtain

$$-v'(l_T^*(1^T)) = \psi_T(i^*)F_2(K_T^*, Y_T^*)u'(c_T^*(1^T)),$$

which concludes the proof of (10).

Given that $j_t^* > 0$ for all $t = 1, ..., T$ and using the fact that (51) is strictly increasing we get that, at the optimum,

$$-v'\left(\frac{\psi_t(0)l_t^*(1^{t-1}, 0)}{\psi_t(j_t^*)}\right)\frac{\psi_t(0)}{\psi_t(j_t^*)} < -v'\left(\frac{\psi_t(0)l_t^*(1^{t-1}, 0)}{\psi_t(0)}\right)\frac{\psi_t(0)}{\psi_t(0)} = -v'\left(l_t^*(0^{t-1})\right),$$

and

$$
\begin{aligned}
-v'\left(\frac{\psi_s(0)l_s^*(1^{t-1}, 0, 0^{s-t})}{\psi_s(j_t^*1(\eta^{s-t}))}\right)\frac{\psi_s(0)}{\psi_s(j_t^*1(\eta^{s-t}))} &< -v'\left(\frac{\psi_s(0)l_s^*(1^{t-1}, 0, 0^{s-t})}{\psi_s(0)}\right)\frac{\psi_s(0)}{\psi_s(0)} \\
&= -v'\left(l_s^*(1^{t-1}, 0, 0^{s-t})\right)
\end{aligned}
$$

for $\eta^{s-t} = 1^{s-t}$. Applying the above inequalities to FO conditions (49)-(50) evaluated at the optimum yields

$$-\beta^t v'(l_t^*(1^{t-1}, 0)) \left[1 + \sum_{s=1}^{t-1} \alpha_s - \alpha_t \frac{\pi^t(1^{t-1}, 1)}{\pi^t(1^{t-1}, 0)}\right] < \psi_t(0)\lambda_t F_2(K_t^*, Y_t^*),$$

$$-\beta^s v'(l_s^*(1^{t-1}, 0, 0^{s-t})) \left[1 + \sum_{n=1}^{t-1} \alpha_n - \alpha_t \sum_{\eta^{s-t}} \frac{\pi^s(1^{t-1}, 1, \eta^{s-t})}{\pi^s(1^{t-1}, 0, 0^{s-t})}\right] < \psi_s(0)\lambda_s F_2(K_s^*, Y_s^*),$$

where the last inequality uses the fact that $\pi^s(1^s) > 0$ for all s. Combining the above with (40)-(47) evaluated at the optimum yields

$$
\begin{aligned}
-v'(l_t^*(1^{t-1}, 0)) &< \psi_t(0)F_2(K_t^*, Y_t^*)u'(c_t^*(1^{t-1}, 0)), \\
-v'(l_s^*(1^{t-1}, 0, 0^{s-t})) &< \psi_s(0)F_2(K_s^*, Y_s^*)u'(c_s^*(1^{t-1}, 0, 0^{s-t})),
\end{aligned}
\tag{53}
$$

for all t, s such that $1 \leq t < s \leq T$, which concludes the proof of (8). $\quad\square$

Proof of Proposition 2

The FO condition with respect to K_{t+1} evaluated at the optimum is given by

$$\frac{\lambda_t}{\lambda_{t+1}} = r^*_{t+1} \tag{54}$$

for each $t = 0, ..., T - 1$, where $r^*_{t+1} := F_1(K^*_{t+1}, Y^*_{t+1}) = Z_1(K^*_{t+1}, Y^*_{t+1}) + 1 - \delta$. Since the low human capital state is absorbing, we have

$$\sum_{\eta^{s-t}} \frac{\pi^s(1^{t-1}, 1, \eta^{s-t})}{\pi^s(1^{t-1}, 0, 0^{s-t})} = \frac{\pi^t(1^{t-1}, 1)}{\pi^t(1^{t-1}, 0)},$$

which allows us to rewrite condition (47) as

$$\beta^s u'(c_s(1^{t-1}, 0, 0^{s-t})) \left[1 + \sum_{n=1}^{t-1} \alpha_n - \alpha_t \frac{\pi^t(1^{t-1}, 1)}{\pi^t(1^{t-1}, 0)} \right] = \lambda_s.$$

The above condition and (54) imply that, at the optimum,

$$\frac{u'(c^*_s(1^{t-1}, 0, 0^{s-t}))}{\beta u'(c^*_{s+1}(1^{t-1}, 0, 0^{s-t+1}))} = \frac{\lambda_s}{\lambda_{s+1}} = r^*_{s+1}, \tag{55}$$

which, given that $\pi^{s+1}(1^{t-1}, 0, 0^{s-t+1})/\pi^s(1^{t-1}, 0, 0^{s-t}) = 1$, can be (trivially) written as

$$\frac{1}{u'(c^*_s(1^{t-1}, 0, 0^{s-t}))} = \frac{1}{\beta r^*_{s+1} u'(c^*_{s+1}(1^{t-1}, 0, 0^{s-t+1}))} = E_{s+1} \left[\frac{1}{\beta r^*_{s+1} u'(c^*_{s+1})} \,\Big|\, \eta^s = (1^{t-1}, 0, 0^{s-t}) \right],$$

which proves (12) for all $s = 1, ..., T - 1$ after all histories η^s such that $h_s(\eta^s) = 0$.

The FO conditions (39) and (40) imply that, for $t = 1, ..., T - 1$, at the optimum,

$$E_t \left[\frac{1}{\beta^{t+1} u'(c^*_{t+1})} \,\Big|\, \eta^t = 1^t \right]$$

$$= \pi_{t+1}(1) \frac{1}{\beta^{t+1} u'(c^*_{t+1}(1^t, 1))} + \pi_{t+1}(0) \frac{1}{\beta^{t+1} u'(c^*_{t+1}(1^t, 0))}$$

$$= \frac{\pi^{t+1}(1^t, 1)}{\pi^t(1^t)} \left[1 + \sum_{s=1}^t \alpha_t + \alpha_{t+1} \right] \lambda_{t+1}^{-1} + \frac{\pi^{t+1}(1^t, 0)}{\pi^t(1^t)} \left[1 + \sum_{s=1}^t \alpha_s - \alpha_{t+1} \frac{\pi^{t+1}(1^t, 1)}{\pi^{t+1}(1^t, 0)} \right] \lambda_{t+1}^{-1}$$

$$= \left[1 + \sum_{s=1}^t \alpha_s \right] \lambda_{t+1}^{-1} = \frac{1}{\beta^t u'(c^*_t(1^t))} \lambda_t \lambda_{t+1}^{-1}$$

Substituting r^*_{t+1} for $\lambda_t \lambda_{t+1}^{-1}$ and dividing through by $\beta^t r^*_{t+1}$, we obtain (12) for all $t = 1, ..., T - 1$

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after histories $\eta^t = 1^t$.

The FO conditions (39) and (40) for $t = 1$ imply that, at the optimum,

$$E\left[\frac{1}{\beta u'(c_1^*)}\right] = \frac{\pi_0(1)}{\beta u'(c_1^*(1))} + \frac{\pi_0(0)}{\beta u'(c_1^*(0))} = \pi_0(1)[1+\alpha_1]\lambda_1^{-1} + \pi_0(0)\left[1-\alpha_1\frac{\pi_0(1)}{\pi_0(0)}\right]\lambda_1^{-1} = \frac{1}{\lambda_1}.$$

Dividing through by r_1^* and using (54) for $t = 0$, we get

$$E\left[\frac{1}{r_1^*\beta u'(c_1^*)}\right] = \frac{1}{\lambda_0}.$$

Using (43) to eliminate λ_0 yields (11), which completes the proof of the proposition. $\square$

Proof of Proposition 3

First, $j_t^* < i^*$ for all t, and the fact that, for each t, ψ_t'/ψ_t is strictly decreasing imply that

$$\frac{\psi_t'(j_t^*)}{\psi_t(j_t^*)}\psi_t(i^*) > \psi_t'(i^*) \tag{56}$$

for all $t \geq 1$.

Writing out the derivative in condition (6), multiplying by negative one, and dropping the terms that have the time index $s > t$, we obtain the following inequality, at the optimum:

$$u_0'(c_0^* + i^* - j_t^*) + \sum_{s=1}^{t-1}\beta^s\pi^s(1^s)v'\left(\frac{\psi_s(i^*)l_s^*(1^s)}{\psi_s(j_t^*)}\right)\frac{l_s^*(1^s)}{\psi_s(j_t^*)}\frac{\psi_s'(j_t^*)}{\psi_s(j_t^*)}\psi_s(i^*) > 0$$

for all $t \geq 1$. Using (56) and the fact that $v' < 0$, we thus get that

$$u_0'(c_0^* + i^* - j_t^*) + \sum_{s=1}^{t-1}\beta^s\pi^s(1^s)v'\left(\frac{\psi_s(i^*)l_s^*(1^s)}{\psi_s(j_t^*)}\right)\frac{l_s^*(1^s)}{\psi_s(j_t^*)}\psi_t'(i^*) > 0 \tag{57}$$

for all $t \geq 1$.

The FO condition with respect to i implies that the optimum satisfies the following condition:

$$\lambda_0 - \sum_{t=1}^{T}\lambda_t\pi^t(1^t)\psi_t'(i^*)l_t^*(1^t)F_2\left(K_t^*, Y_t^*\right)$$
$$= -\sum_{t=1}^{T}\alpha_t u_0'(c_0^* + i^* - j_t^*) - \sum_{t=2}^{T}\alpha_t\left[\sum_{s=1}^{t-1}\beta^s\pi^s(1^s)v'\left(\frac{\psi_s(i^*)l_s^*(1^s)}{\psi_s(j_t^*)}\right)\frac{\psi_s'(i^*)l_s^*(1^s)}{\psi_s(j_t^*)}\right].$$

Using (57) for $t = 2, ..., T$ and the fact that all $\alpha_t > 0$, as well as the fact that $u_0'(c_0^* + i^* - j_1^*) > 0$,

we get that the right-hand side of the above condition is strictly negative. Thus, we have

$$\lambda_0 < \sum_{t=1}^{T} \lambda_t R_t^*,$$

where $R_t^* = \pi^t(1^t)\psi_t'(i^*)l_t^*(1^t)F_2\left(K_t^*, Y_t^*\right)$. Given that $\lambda_t/\lambda_0 = \prod_{s=1}^{t} 1/r_s^*$ we obtain the conclusion. $\quad\square$

Proof of Theorem 1

We prove this proposition in two steps. In step 1 we show that there exists an optimal tax system $\tilde{\mathcal{T}}_0^*$, $\{\mathcal{T}_t^*, \phi_t^*\}_{t=1}^{T}$. In step 2 we show the alleged properties of the optimal capital taxes.

Step 1 In order to show that there exists an optimal tax system $\tilde{\mathcal{T}}_0^*$, $\{\mathcal{T}_t^*, \phi_t^*\}_{t=1}^{T}$, we need to show that if taxes are $\tilde{\mathcal{T}}_0^*$, $\{\mathcal{T}_t^*, \phi_t^*\}_{t=1}^{T}$, then the equilibrium allocation $(c^e, i^e, k^e, l^e, K^e, Y^e)$ is such that

$$(c^e, i^e, l^e, K^e, Y^e) = (c^*, i^*, l^*, K^*, Y^*), \tag{58}$$

where $(c^*, i^*, l^*, K^*, Y^*)$ is (a part of) the optimal allocation. Let $\hat{k}^* = \{\hat{k}_t^*\}_{t=1}^{T}$, $\hat{k}_t^* : \Theta^{t-1} \to \mathbb{R}_+$ be a process of individual capital holdings that is consistent with the optimal aggregate capital sequence $K^* = \{K_t^*\}_{t=1}^{T}$, i.e., such that

$$\sum_{\eta^t \in \Theta^t} \pi^t(\eta^t)\hat{k}_{t+1}^*(\eta^t) = K_{t+1}^* \tag{59}$$

for all $t = 0, ..., T-1$, where, as before, η^0 denotes the empty history.

We now show that, for a fixed distribution of capital holdings $\hat{k}^*$, the following reduced-form tax system $\tilde{\mathcal{T}}_0^*$, $\{\mathcal{T}_t^*, \phi_t^*\}_{t=1}^{T}$ implements the optimum:

$$\tau_{1,1}^*(0) \quad = \quad 1 - \frac{u_0'(c_0^* + i^* - j_1^*)}{r_1^*\beta u'(c_1^*(0))},$$

$$\tau_{1,t}^*(1^{t-1}, 0) \quad = \quad \frac{u_0'(c_0^*) - u_0'(c_0^* + i^* - j_t^*) + \sum_{s=t+1}^{T}\left[\sum_{n=t+1}^{s}\prod_{i=n}^{s}\frac{\pi^i(1^{i-1}, 0)}{\pi^i(1^{i-1}, 1)}\right]\left[u_0'(c_0^*) - u_0'(c_0^* + i^* - j_s^*)\right]}{\pi^t(1^t)r_1^*\beta^t u'(c_t^*(1^{t-1}, 0))} \quad \text{for } t > 1,$$

$$\tau_{1,1}^*(1) \quad = \quad 1 - \frac{u_0'(c_0^* + i^* - j_2^*) - \pi^1(0)u_0'(c_0^* + i^* - j_1^*) + \pi^1(1)r_1^*\beta^2 u'(c_2^*(1,0))\tau_{1,2}^*(1,0)}{\pi^1(1)r_1^*\beta u'(c_1^*(1))},$$

$$\tau_{1,t}^*(\eta^t) \quad = \quad 0 \ \text{ for } \eta^t \neq (1^{t-1}, 0), \ t > 1,$$

$$\tau_{t,t}^*(\eta^t) \quad = \quad 1 - \frac{u'(c_{t-1}^*(\eta^{t-1}))}{r_t^*\beta u'(c_t^*(\eta^t))} \ \text{ for } t > 1,$$

$$\tau_{t,s}^*(\eta^t) \quad = \quad 0 \ \text{ for } t < s \leq T,$$

and

$$\phi_1^*(\theta) \quad = \quad w_1^* y_1^*(\theta) + (1 - \tau_{1,1}(\theta)) r_1^* \hat{k}_1^* - c_1^*(\theta) - \hat{k}_2^*(\theta), \tag{60}$$

for $\theta \in \Theta$, and, for $t = 2, ...T$, $\eta^t \in \Theta^t$,

$$\phi_t^*(\eta^t) \quad = \quad w_t^* y_t^*(\eta^t) + (1 - \tau_{t,t}(\eta^t)) r_t^* \hat{k}_t^*(\eta^{t-1}) - \tau_{1,t}^*(\eta^t) r_1^* \hat{k}_1^* - c_t^*(\eta^t) - \hat{k}_{t+1}^*(\eta^t), \tag{61}$$

where

$$r_t^* \quad = \quad F_1(K_t^*, Y_t^*), \tag{62}$$

$$w_t^* \quad = \quad F_2(K_t^*, Y_t^*). \tag{63}$$

We need to show that our candidate equilibrium allocation $(c^*, i^*, \hat{k}^*, l^*, K^*, Y^*)$ (a) is consistent with competitive pricing conditions at prices (r^*, w^*), (b) satisfies market clearing, and (c) under the tax system given above, is consistent with agent's utility maximization. Directly from (62) and (63) we get the competitive pricing conditions. By resource feasibility of the optimal allocation A^* and (59), we have that the candidate equilibrium allocation $(c^*, i^*, \hat{k}^*, l^*, K^*, Y^*)$ satisfies market clearing. All that remains to be shown is that $(c^*, i^*, \hat{k}^*, l^*)$ is individually optimal for the agents, given taxes $(\mathcal{T}^*, \phi^*)$ and prices (r^*, w^*).

A labor effort strategy $l = (l_1, ..., l_T)$ uniquely determines an effective labor supply process $y = (y_1, ..., y_T)$ through the productivity function $y_t(\eta^t) = \psi_t (h_t(\eta^t) l_t(\eta^t))$. Due to the fact that all reduced-form tax mechanisms severely punish observed effective labor supply paths $y^t \notin \{y^{*t}(\eta^t)\}_{\eta^t \in \Theta^t}$, under a reduced-form tax system the agent's effective labor effort strategy l must be such that there exists a measurable individual shock announcement strategy $\zeta : \Theta^T \to \Theta^T$ such that, for all $t = 1, ..., T$ and all $\eta^t \in \Theta^t$

$$y^t(\eta^t) = y^{*t}(\zeta(\eta^t)),$$

where $y^{*t} = (y_1^*, ..., y_t^*)$. Let $\mathcal{Z}$ denote the (finite) set of all measurable individual shock announcement strategies ζ. Under a reduced-form tax system, agents' utility maximization problem is indirectly (through the restriction on the observable effective labor supply paths y^T) reduced to the choice of $\zeta \in \mathcal{Z}$ and (c, i, k). Conditional on ζ, under taxes $(\mathcal{T}^*, \phi^*)$ and prices (r^*, w^*), agents' optimal choices of (c, i, k) solve the following problem:

$$\max_{c,i,k} \quad u_0(c_0) + \sum_{t=1}^{T} \sum_{\eta^t \in \Theta^t} \beta^t \pi^t(\eta^t) \left[u(c_t(\eta^t)) + v \left(\frac{y_t^*(\zeta(\eta^t))}{\psi_t(i1(\eta^t))} \right) \right]$$

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subject to

$$c_0 + i - k_1 \leq k_0, \tag{64}$$

$$c_1(\theta) + k_2(\theta) \leq w_1^* y_1^*(\zeta(\theta)) + (1 - \tau_{1,1}^*(\zeta(\theta)))r_1^* k_1 - \phi_1^*(\theta) \tag{65}$$

for $\theta \in \Theta$, and, for $t = 2,...T$, $\eta^t \in \Theta^t$,

$$c_t(\eta^t) + k_{t+1}(\eta^t) \leq w_t^* y_t^*(\zeta(\eta^t)) + (1 - \tau_{t,t}^*(\zeta(\eta^t)))r_t^* k_t(\eta^{t-1}) - \phi_t^*(\zeta(\eta^t)) - \tau_{1,t}^*(\zeta(\eta^t)))r_1^* k_1. \tag{66}$$

Since capital income taxes are linear, the budget constraints in this problem are linear. Due to strict concavity of the preferences and the convexity of the constraint set, there is a unique solution to this problem. Due to the assumed Inada conditions, the solution is interior. The first-order conditions to this problem, with respect to i, k_1 and $k_{t+1}(\eta^t)$ for $t = 1,...T - 1$, $\eta^t \in \Theta^t$, are as follows

$$u_0'(c_0) = -\sum_{t=1}^{T} \beta^t \pi^t(1^t) v'\left(\frac{y_t^*(\zeta(1^t))}{\psi_t(i)}\right) \frac{y_t^*(\zeta(1^t))}{\psi_t^2(i)} \psi_t'(i), \tag{67}$$

$$u_0'(c_0) = \beta r_1^* E\left[\left(1 - \tau_{1,1}^*(\zeta)\right) u'(c_1)\right] + \sum_{t=2}^{T} \beta^t r_1^* E\left[\tau_{1,t}^*(\zeta) u'(c_t)\right], \tag{68}$$

$$u'(c_t(\eta^t)) = \beta r_{t+1}^* E_t\left[\left(1 - \tau_{t+1,t+1}^*(\zeta)\right) u'(c_{t+1}) \,|\, \eta^t\right], \tag{69}$$

where, for $1 \leq s \leq t \leq T$, $\tau_{s,t}^*(\zeta)(\cdot) = \tau_{s,t}^*(\zeta(\cdot))$. Combined with the budget constraints (64)–(66) written as equalities, these FO conditions are both necessary and sufficient for the optimum. Let $\hat{c}(\zeta)$, $\hat{i}(\zeta)$, and $\hat{k}(\zeta)$ denote the solution to this problem, for a given $\zeta \in \mathcal{Z}$.

Among all reporting strategies in $\mathcal{Z}$, let ζ^* denote truth-telling, i.e., $\zeta^*(\eta^t) = \eta^t$ for all η^t. Also, for $t = 1,...,T$, let ζ^t denote the reporting strategy that corresponds to shirking in period t, i.e., $\zeta^t(\eta^{t-1}) = \eta^{t-1}$ for all η^{t-1}, and $\zeta^t(\eta^{t-1}, \eta^{s-t+1}) = (\eta^{t-1}, 0^{s-t+1})$ for all η^{t-1}, $s = t,...,T$, all $\eta^{s-t+1} \in \Theta^{s-t+1}$. Note that these T shirking strategies represent the binding IC constraints in the direct revelation mechanism.

We claim that, under the proposed tax system, for each of these $T+1$ strategies, agents' optimal choices in the tax mechanism exactly replicate the allocation of human capital investment and consumption that these strategies yield in the direct revelation mechanism. That is, we claim that

$$\hat{i}(\zeta^*) = i^*,$$
$$\hat{c}(\zeta^*) = c^*,$$

and, for all $t = 1, ..., T$,

$$
\begin{aligned}
\hat{\imath}(\zeta^t) &= j_t^*, \\
\hat{c}_0(\zeta^t) &= c_0^* + i^* - j_t^*, \\
\hat{c}_s(\zeta^t)(\eta^s) &= c_s^*(\zeta^t(\eta^s))
\end{aligned}
$$

for all $s = 1, ..., T$, $\eta^s \in \Theta^s$.

To show that this claim is correct, it is enough to demonstrate that the proposed solutions to the conditional utility maximization problem, together with some capital holding plans $\hat{k}(\zeta)$ for $\zeta = \zeta^*, \zeta^1, ..., \zeta^T$ satisfy the FO conditions and budget constraints, which are sufficient for the maximum.

Indeed, with capital holding plans given by

$$
\begin{aligned}
\hat{k}_{s+1}(\zeta^*)(\eta^s) &= \hat{k}_{s+1}^*(\eta^s), \\
\hat{k}_{s+1}(\zeta^t)(\eta^s) &= \hat{k}_{s+1}^*(\zeta^t(\eta^s))
\end{aligned}
$$

for all $s = 1, ..., T$, $\eta^s \in \Theta^s$, we use (60) and (61) to check that the proposed conditional solutions satisfy the budget constraints (64)–(66). Also, we use (7) and (6) to check that the proposed conditional solutions satisfy the FO conditions with respect to i, i.e., (67). All that remains to be checked, therefore, are the Euler equations (68) and (69). Substituting the formulas for the proposed marginal tax rates $\tau_{s,t}^*$, after some algebra, we get that this is true, which proves our claim.

By the above claim, each of the T shirking strategies, as well as truth-telling, yields the same amount of utility in the market mechanism as it does in the direct revelation mechanism. By incentive compatibility of the optimum, none of the shirking strategies upsets the truth-telling in the market mechanism.

Our proof is complete if none of the remaining reporting strategies $\zeta \neq \zeta^*, \zeta^1, ..., \zeta^T$ yields in the market mechanism more utility than does truth-telling, which indeed is true when $\{\psi_t(0)\}_{t=1}^T$ is small enough, i.e., in the generic case. This follows from the fact that each of these strategies involves an "upward lie" in some contingency, i.e., calls for the supply of a high amount of effective labor $y_t > 0$ at a near-zero skill level $\psi_t \cong 0$, which requires an exploding level of effort l_t and leads to very large disutility $-v(l_t)$, i.e., cannot be individually optimal.

Step 2 To complete the proof of the proposition we need to demonstrate that the signs of the marginal tax rates are indeed as specified in the statement of the proposition. First, we note that

for $t = 2, ..., T$:

$$\tau_{1,t}^*(1^{t-1}, 0) =$$

$$\frac{u_0'(c_0^*) - u_0'(c_0^* + i^* - j_t^*) + \sum_{s=t+1}^{T} \left[\sum_{n=t+1}^{s} \prod_{i=n}^{s} \frac{\pi^i(1^{i-1}, 0)}{\pi^i(1^{i-1}, 1)} \right] [u_0'(c_0^*) - u_0'(c_0^* + i^* - j_s^*)]}{\pi^t(1^t) r_1^* \beta^t u'(c_t^*(1^{t-1}, 0))} > 0, \qquad (70)$$

because $u' > 0$, u_0' is strictly decreasing and $i^* > j_t^*$, $\alpha_t > 0$ for $t \geq 1$.

Second, we observe that the FO conditions with respect to c_0, $c_1(0)$, and K_1 characterizing the optimum imply that

$$r_1^* \beta u'(c_1^*(0)) = \frac{u_0'(c_0^*) + \sum_{t=1}^{T} \alpha_t [u_0'(c_0^*) - u_0'(c_0^* + i^* - j_t^*)]}{\left[1 - \alpha_1 \frac{\pi^1(1)}{\pi^1(0)} \right]} >$$

$$u_0'(c_0^*) + \sum_{t=1}^{T} \alpha_t [u_0'(c_0^*) - u_0'(c_0^* + i^* - j_t^*)] > u_0'(c_0^*) > u_0'(c_0^* + i^* - j_1^*),$$

where the first inequality follows from the fact that $0 < \left[1 - \alpha_1 \frac{\pi^1(1)}{\pi^1(0)} \right] < 1^{16}$, while the second and the third one follow again from the fact that u_0' is strictly decreasing and $i^* > j_t^*$, $\alpha_t > 0$ for $t \geq 1$. But this implies that

$$1 > \frac{u_0'(c_0^* + i^* - j_1^*)}{r_1^* \beta u'(c_1^*(0))} = 1 - \tau_1^*(0) \Rightarrow \tau_1^*(0) > 0. \qquad (71)$$

Third, we observe that

$$1 - \tau_{1,1}^*(1) = \frac{u_0'(c_0^* + i^* - j_2^*) - \pi^1(0)u_0'(c_0^* + i^* - j_1^*) + \pi^1(1)r_1^* \beta^2 u'(c_2^*(1,0))\tau_{1,2}^*(1,0)}{\pi^1(1)r_1^* \beta u'(c_1^*(1))} >$$

$$\frac{u_0'(c_0^* + i^* - j_2^*) - \pi^1(0)u_0'(c_0^* + i^* - j_1^*)}{\pi^1(1)r_1^* \beta u'(c_1^*(1))} >$$

$$\frac{u_0'(c_0^* + i^* - j_1^*) - \pi^1(0)u_0'(c_0^* + i^* - j_1^*)}{\pi^1(1)r_1^* \beta u'(c_1^*(1))} = \frac{u_0'(c_0^* + i^* - j_1^*)}{r_1^* \beta u'(c_1^*(1))} >$$

$$\frac{u_0'(c_0^* + i^* - j_1^*)}{r_1^* \beta u'(c_1^*(0))} = 1 - \tau_{1,1}^*(0),$$

where the first of inequalities follows from $\tau_{1,2}^*(1,0) > 0$, the second one from $u_0'' < 0$ and the fact that $j_2^* > j_1^*$, and the third one from from $u'' < 0$ and the fact that $c_1^*(1) > c_1^*(0)$. The above implies that $\tau_{1,1}^*(1) < \tau_{1,1}^*(0)$, which combined with (78), (70), and (71) yields

$$\tau_{1,1}^*(0) > 0 > \tau_{1,1}^*(1).$$

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[16] This follows from (40) and the fact that $\alpha_1 > 0$.

Next, we observe that for $t = 2, ..., T$:

$$1 - \tau^*_{t,t}(1^{t-1}, 0) = \frac{u'(c_{t-1}(1^{t-1}))}{r^*_{t+1}\beta u'(c^*_t(1^{t-1}, 0))} < \frac{u'(c_{t-1}(1^{t-1}))}{r^*_{t+1}\beta u'(c^*_t(1^{t-1}, 1))} = 1 - \tau^*_{t,t}(1^{t-1}, 0),$$

where the inequality follows from the fact that u' is strictly decreasing and $c^*_t(1^{t-1}, 1) > c^*_t(1^{t-1}, 0)$. Combing the above with (80) yields

$$\tau^*_{t,t}(1^{t-1}, 0) > 0 > \tau^*_{t,t}(1^{t-1}, 1) \quad \text{for } t = 2, ..., T.$$

Finally, (55) directly implies that $\tau^*_{t,t}(1^{t-s}, 0^s) = 0$ for all $1 < s \le t \le T$, all η^t. $\quad\square$

Proof of Proposition 4

First, we will show that

$$E\left[\tau^*_1 + \sum_{t=2}^T \left(\prod_{s=2}^t \frac{1}{r^*_s}\right) \tau^*_{1,t}\right] = 0$$

Using the formulas for the optimal taxes we find that

$$E\left[\tau^*_1 + \sum_{t=2}^T \left(\prod_{s=2}^t \frac{1}{r^*_s}\right) \tau^*_{1,t}\right] = 1 - \left\{ \frac{\pi^1(0)u'_0(c^*_0 + i^* - j^*_1)}{r^*_1\beta u'(c^*_1(0))} + \frac{u'_0(c^*_0 + i^* - j^*_2) - \pi^1(0)u'_0(c^*_0 + i^* - j^*_1)}{r^*_1\beta u'(c_1(1))}\right.$$

$$+ \frac{\pi^1(1)}{\pi^2(1^2)} \frac{\left[u'_0(c^*_0) - u'_0(c^*_0 + i^* - j^*_2) + \sum_{s=3}^T \left[\sum_{n=3}^s \prod_{i=n}^s \frac{\pi^i(1^{i-1},0)}{\pi^i(1^{i-1},1)}\right][u'_0(c^*_0) - u'_0(c^*_0 + i^* - j^*_s)]\right]}{r^*_1\beta u'(c_1(1))}$$

$$- \sum_{t=2}^T \left(\prod_{s=2}^t \frac{1}{r^*_s}\right) \frac{\pi^t(1^{t-1},0)}{\pi^t(1^{t-1},1)} \left(\frac{u'_0(c^*_0) - u'_0(c^*_0 + i^* - j^*_t)}{r^*_1\beta^t u'(c^*_t(1^{t-1},0))}\right)$$

$$\left. - \sum_{t=2}^T \left(\prod_{s=2}^t \frac{1}{r^*_s}\right) \frac{\pi^t(1^{t-1},0)}{\pi^t(1^{t-1},1)} \left(\frac{\sum_{s=t+1}^T \left[\sum_{n=t+1}^s \prod_{i=n}^s \frac{\pi^i(1^{i-1},0)}{\pi^i(1^{i-1},1)}\right][u'_0(c^*_0) - u'_0(c^*_0 + i^* - j^*_s)]}{r^*_1\beta^t u'(c^*_t(1^{t-1},0))}\right)\right\} \quad (72)$$

The FO conditions of the optimum with respect to $c_t(1)$, $c_t(1^{t-1}, 0)$, and K_t imply that

$$u'(c^*_1(1^1)) = \left(\frac{1 - \alpha_1 \frac{\pi^1(1)}{\pi^1(0)}}{1 + \alpha_1}\right) u'(c^*_1(0)), \quad (73)$$

$$\beta^t u'(c^*_t(1^{t-1}, 0)) = \left(\frac{1 - \alpha_1 \frac{\pi^1(1)}{\pi^1(0)}}{1 + \sum_{s=1}^{t-1} \alpha_s - \alpha_t \frac{\pi^t(1^{t-1},1)}{\pi^t(1^{t-1},0)}}\right) \left(\prod_{s=2}^t \frac{1}{r^*_s}\right) \beta u'(c^*_1(1, 0)). \quad (74)$$

Using the above conditions allows us to rewrite (72) as

$$E\left[\tau_1^* + \sum_{t=2}^{T}\left(\prod_{s=2}^{t}\frac{1}{r_s^*}\right)\tau_{1,t}^*\right] = 1 - \frac{1}{r_1^*\beta u'(c_1^*(0))\left[1-\alpha_1\frac{\pi^1(1)}{\pi^1(0)}\right]}\left\{\left[1-\alpha_1\frac{\pi^1(1)}{\pi^1(0)}\right]\pi^1(0)u_0'(c_0^*+i^*-j_1^*)\right.$$

$$+[1+\alpha_1]u_0'(c_0^*+i^*-j_2^*)-[1+\alpha_1]\pi^1(0)u_0'(c_0^*+i^*-j_1^*)$$

$$+\frac{\pi^1(1)}{\pi^2(1^2)}[1+\alpha_1]\left[u_0'(c_0^*)-u_0'(c_0^*+i^*-j_2^*)+\sum_{s=3}^{T}\left[\sum_{n=3}^{s}\prod_{i=n}^{s}\frac{\pi^i(1^{i-1},0)}{\pi^i(1^{i-1},1)}\right][u_0'(c_0^*)-u_0'(c_0^*+i^*-j_s^*)]\right]$$

$$-\sum_{t=2}^{T}\left\{\left(\frac{\pi^t(1^{t-1},0)}{\pi^t(1^{t-1},1)}\left[1+\sum_{k=1}^{t-1}\alpha_j\right]-\alpha_t\right)\times\right.$$

$$\left.\left(u_0'(c_0^*)-u_0'(c_0^*+i^*-j_t^*)+\sum_{s=t+1}^{T}\left[\sum_{n=t+1}^{s}\prod_{i=n}^{s}\frac{\pi^i(1^{i-1},0)}{\pi^i(1^{i-1},1)}\right][u_0'(c_0^*)-u_0'(c_0^*+i^*-j_s^*)]\right)\right\}\right\}$$

We note that in the above formula the multiplier of the term $u_0'(c_0^*)-u_0'(c_0^*+i^*-j_s^*)$ for $s\geq 3$ is equal to

$$[1+\alpha_1]\frac{\pi^1(1)}{\pi^2(1^2)}\left(\sum_{n=3}^{s}\prod_{i=n}^{s}\frac{\pi^i(1^{i-1},0)}{\pi^i(1^{i-1},1)}\right)$$

$$-\sum_{t=2}^{s-1}\left\{\left(\frac{\pi^t(1^{t-1},0)}{\pi^t(1^{t-1},1)}\left[1+\sum_{k=1}^{t-1}\alpha_j\right]-\alpha_t\right)\left(\sum_{n=t+1}^{s}\prod_{i=n}^{s}\frac{\pi^i(1^{i-1},0)}{\pi^i(1^{i-1},1)}\right)\right\}$$

$$-\frac{\pi^s(1^{s-1},0)}{\pi^s(1^{s-1},1)}\left[1+\sum_{k=1}^{s-1}\alpha_k\right]+\alpha_s. \tag{75}$$

Noting that

$$\sum_{t=2}^{s-1}\left\{\left(\frac{\pi^t(1^{t-1},0)}{\pi^t(1^{t-1},1)}\left[1+\sum_{k=1}^{t-1}\alpha_j\right]-\alpha_t\right)\left(\sum_{n=t+1}^{s}\prod_{i=n}^{s}\frac{\pi^i(1^{i-1},0)}{\pi^i(1^{i-1},1)}\right)\right\}+\frac{\pi^s(1^{s-1},0)}{\pi^s(1^{s-1},1)}\left[1+\sum_{k=1}^{s-1}\alpha_k\right]=$$

$$[1+\alpha_1]\sum_{t=2}^{s-1}\left\{\left(\frac{\pi^t(1^{t-1},0)}{\pi^t(1^{t-1},1)}\right)\left(\sum_{n=t+1}^{s}\prod_{i=n}^{s}\frac{\pi^i(1^{i-1},0)}{\pi^i(1^{i-1},1)}\right)\right\}+[1+\alpha_1]\frac{\pi^s(1^{s-1},0)}{\pi^s(1^{s-1},1)},$$

allows us to rewrite (75) as

$$[1+\alpha_1]\frac{\pi^1(1)}{\pi^2(1^2)}\left(\sum_{n=3}^{s}\prod_{i=n}^{s}\frac{\pi^i(1^{i-1},0)}{\pi^i(1^{i-1},1)}\right)$$

$$-[1+\alpha_1]\sum_{t=2}^{s-1}\left\{\left(\frac{\pi^t(1^{t-1},0)}{\pi^t(1^{t-1},1)}\right)\left(\sum_{n=t+1}^{s}\prod_{i=n}^{s}\frac{\pi^i(1^{i-1},0)}{\pi^i(1^{i-1},1)}\right)\right\}-[1+\alpha_1]\frac{\pi^s(1^{s-1},0)}{\pi^s(1^{s-1},1)}+\alpha_s. \tag{76}$$

Noting further that

$$\frac{\pi^1(1)}{\pi^2(1^2)}\left(\sum_{n=3}^{s}\prod_{i=n}^{s}\frac{\pi^i(1^{i-1},0)}{\pi^i(1^{i-1},1)}\right)=\sum_{t=2}^{s-1}\left\{\left(\frac{\pi^t(1^{t-1},0)}{\pi^t(1^{t-1},1)}\right)\left(\sum_{n=t+1}^{s}\prod_{i=n}^{s}\frac{\pi^i(1^{i-1},0)}{\pi^i(1^{i-1},1)}\right)\right\}+\frac{\pi^s(1^{s-1},0)}{\pi^s(1^{s-1},1)},$$

and using this in (76) implies that the multiplier of the term $u_0'(c_0^*) - u_0'(c_0^* + i^* - j_s^*)$ for $s \geq 3$ in (72) is equal to α_s. But this implies that

$$E\left[\tau_1^* + \sum_{t=2}^{T}\left(\prod_{s=2}^{t}\frac{1}{r_s^*}\right)\tau_{1,t}^*\right] = 1 - \frac{1}{r_1^*\beta u'(c_1^*(0))\left[1 - \alpha_1\frac{\pi^1(1)}{\pi^1(0)}\right]}\left\{\left[1 - \alpha_1\frac{\pi^1(1)}{\pi^1(0)}\right]\pi^1(0)u_0'(c_0^* + i^* - j_1^*)\right.$$

$$+ [1+\alpha_1]u_0'(c_0^* + i^* - j_2^*) - [1+\alpha_1]\pi^1(0)u_0'(c_0^* + i^* - j_1^*)$$

$$+ \frac{\pi^1(1)}{\pi^2(1^2)}[1+\alpha_1]\left[u_0'(c_0^*) - u_0'(c_0^* + i^* - j_2^*)\right] - \left(\frac{\pi^2(1,0)}{\pi^2(1,1)}[1+\alpha_1] - \alpha_2\right) \times [u_0'(c_0^*) - u_0'(c_0^* + i^* - j_2^*)]$$

$$\left. + I_{T>3}\sum_{t=3}^{T}\alpha_t\left[u_0'(c_0^*) - u_0'(c_0^* + i^* - j_t^*)\right]\right\}$$

Combining the common terms yields

$$E\left[\tau_1^* + \sum_{t=2}^{T}\left(\prod_{s=2}^{t}\frac{1}{r_s^*}\right)\tau_{1,t}^*\right] = 1 - \frac{1}{r_1^*\beta u'(c_1^*(0))\left[1 - \alpha_1\frac{\pi^1(1)}{\pi^1(0)}\right]}\left\{u_0'(c_0^*)(1 + \alpha_1 + \alpha_2)\right.$$

$$\left. - \alpha_1 u_0'(c_0^* + i^* - j_1^*) - \alpha_2 u_0'(c_0^* + i^* - j_2^*) + I_{T>3}\sum_{t=3}^{T}\alpha_t\left[u_0'(c_0^*) - u_0'(c_0^* + i^* - j_t^*)\right]\right\},$$

which is equivalent to

$$E\left[\tau_1^* + \sum_{t=2}^{T}\left(\prod_{s=2}^{t}\frac{1}{r_s^*}\right)\tau_{1,t}^*\right] = 1 - \left(\frac{u_0'(c_0^*) + \sum_{t=1}^{T}\alpha_t\left[u_0'(c_0^*) - u_0'(c_0^* + i^* - j_t^*)\right]}{r_1^*\beta u'(c_1^*(0))\left[1 - \alpha_1\frac{\pi^1(1)}{\pi^1(0)}\right]}\right) \tag{77}$$

Using (73) we find that

$$E\left(\frac{1}{r_1^*\beta u'(c_1^*)}\right) = \frac{1}{r_1^*\beta}\left(\frac{\pi(0)\left(1 - \alpha_1\frac{\pi^1(1)}{\pi^1(0)}\right) + \pi(1)(1 + \alpha_1)}{u'(c_1^*(0)))\left(1 - \alpha_1\frac{\pi^1(1)}{\pi^1(0)}\right)}\right) = \frac{1}{r_1^*\beta u'(c_1^*(0))\left[1 - \alpha_1\frac{\pi^1(1)}{\pi^1(0)}\right]}$$

which allows us to rewrite (77) as

$$E\left[\tau_1^* + \sum_{t=2}^{T}\left(\prod_{s=2}^{t}\frac{1}{r_s^*}\right)\tau_{1,t}^*\right] = 1 - \left(\frac{u_0'(c_0^*) + \sum_{t=1}^{T}\alpha_t\left[u_0'(c_0^*) - u_0'(c_0^* + i^* - j_t^*)\right]}{E\left(\frac{1}{r_1^*\beta u'(c_1^*))}\right)}\right)$$

The modified Rogerson condition for physical capital (Proposition 2) implies that

$$\left(\frac{u_0'(c_0^*) + \sum_{t=1}^{T} \alpha_t \left[u_0'(c_0^*) - u_0'(c_0^* + i^* - j_t^*) \right]}{E\left(\frac{1}{r_1^* \beta u'(c_1^*))} \right)} \right) = 1,$$

which implies that

$$E\left[\tau_1^* + \sum_{t=2}^{T} \left(\prod_{s=2}^{t} \frac{1}{r_s^*} \right) \tau_{1,t}^* \right] = 0. \tag{78}$$

To conclude the proof of part (i) of the proposition we need to show that

$$E[\tau_{t,s}^* \,|\, \eta^{s-1}] = 0, \text{ for any } 1 < t \le s, \, \eta^{s-1}$$

First, when $1 < t < s$, we have that $\tau_{t,s}^*(\eta^s) = 0$, which trivially implies that

$$E[\tau_{t,s}^* \,|\, \eta^{s-1}] = 0, \text{ for any } 1 < t < s, \, \eta^{s-1} \tag{79}$$

When $1 < t = s$, substituting for the optimal capital taxes we find that

$$E[\tau_{t,t}^* \,|\, \eta^{t-1}] = E\left[1 - \frac{u'(c_t^*)}{\beta r_{t+1}^* u'(c_{t+1}^*)} \,\bigg|\, \eta^{t-1} \right].$$

The Rogerson's intertemporal conditions that hold for $t > 1$ at the optimum (Proposition 2) imply that $E\left[1 - \frac{u'(c_t^*)}{\beta r_{t+1}^* u'(c_{t+1}^*)} \,\big|\, \eta^{t-1} \right] = 0$, which yields

$$E[\tau_{t,t}^* \,|\, \eta^{t-1}] = 0, \text{ for any } t > 1, \, \eta^{t-1}. \tag{80}$$

□

Proof of Proposition 5

First, we show that $r_1^* > \hat{r}_1$. Using (in this order) the modified Rogerson condition (11), the facts that, for $t = 1, .., T$, $i^* > j_t^*$ and $\alpha_t > 0$, the assumption $\hat{c} = c^*$, and the standard Rogerson condition (24) of the exogenous-skill economy, we get

$$r_1^* = E\left[\frac{u_0'(c_0^*) + \sum_{t=1}^{T} \alpha_t \left[u_0'(c_0^*) - u_0'(c_0^* + i^* - j_t^*) \right]}{\beta u'(c_1^*)} \right] > E\left[\frac{u_0'(c_0^*)}{\beta u'(c_1^*)} \right] = E\left[\frac{u_0'(\hat{c}_0)}{\beta u'(\hat{c}_1)} \right] = \hat{r}_1.$$

That the intertemporal wedge satisfies $\omega_0^* > \hat{\omega}_0$ follows immediately from $r_1^* > \hat{r}_1$ and the assumption $c^* = \hat{c}$:

$$\omega_0^* = 1 - \frac{u_0'(c_0^*)}{r_1^*\beta E[u'(c_1^*)]} > 1 - \frac{u_0'(\hat{c}_0)}{\hat{r}_1\beta E[u'(\hat{c}_1)]} = \hat{\omega}_0.$$

Moving on to the volatility of marginal tax rates at $t = 1$, we note that, in the exogenous-skill economy, the Rogerson condition (24) at $t = 0$ immediately implies that

$$E\left[\hat{\tau}_1\right] = 0,$$

where $\hat{\tau}_1$ are given in (25). Since $\hat{c}_1 = c_1^*$, and we have a positive spread of consumption at the optimum c^*, we also have

$$\hat{c}_1(1) > \hat{c}_1(0),$$

which implies that

$$\hat{\tau}_1(0) > 0 > \hat{\tau}_1(1).$$

In the endogenous-skill economy, the total marginal tax rate on capital income $r_1 k_1$ is given by

$$\tau_{1,1}^* + \sum_{t=2}^{T}\left(\prod_{s=2}^{t}\frac{1}{r_s^*}\right)\tau_{1,t}^*.$$

By (19) and (21), we have

$$\tau_{1,1}^*(0) + E_1\left[\left.\sum_{t=2}^{T}\left(\prod_{s=2}^{t}\frac{1}{r_s^*}\right)\tau_{1,t}^*\right| \theta = 0\right] = \tau_{1,1}^*(0) < 0,$$

i.e., the total marginal tax rate conditional on $\theta = 0$ is negative. The zero expected total tax result (22) implies then that

$$\tau_{1,1}^*(1) + E_1\left[\left.\sum_{t=2}^{T}\left(\prod_{s=2}^{t}\frac{1}{r_s^*}\right)\tau_{1,t}^*\right| \theta = 1\right] > 0.$$

In both economies, therefore, total marginal tax rates on $r_1 k_1$ are zero in expectation and negative conditional on $\theta = 0$. Therefore, the variance of the total marginal tax rate on $r_1 k_1$ is larger in the endogenous-skill economy iff

$$\tau_{1,1}^*(0) < \hat{\tau}_1(0).$$

Using $i^* > j_1^*$, $r_1^* > \hat{r}_1$ and $\hat{c} = c^*$, we get

$$\tau_1^*(0) = 1 - \frac{u_0'(c_0^* + i^* - j_1^*)}{r^*\beta u'(c_1^*(0))} > 1 - \frac{u_0'(c_0^*)}{r_1^*\beta u'(c_1^*(0))} > 1 - \frac{u_0'(\hat{c}_0)}{\hat{r}_1\beta u'(\hat{c}_1(0))} = \hat{\tau}_1(0),$$

which completes the proof. $\square$

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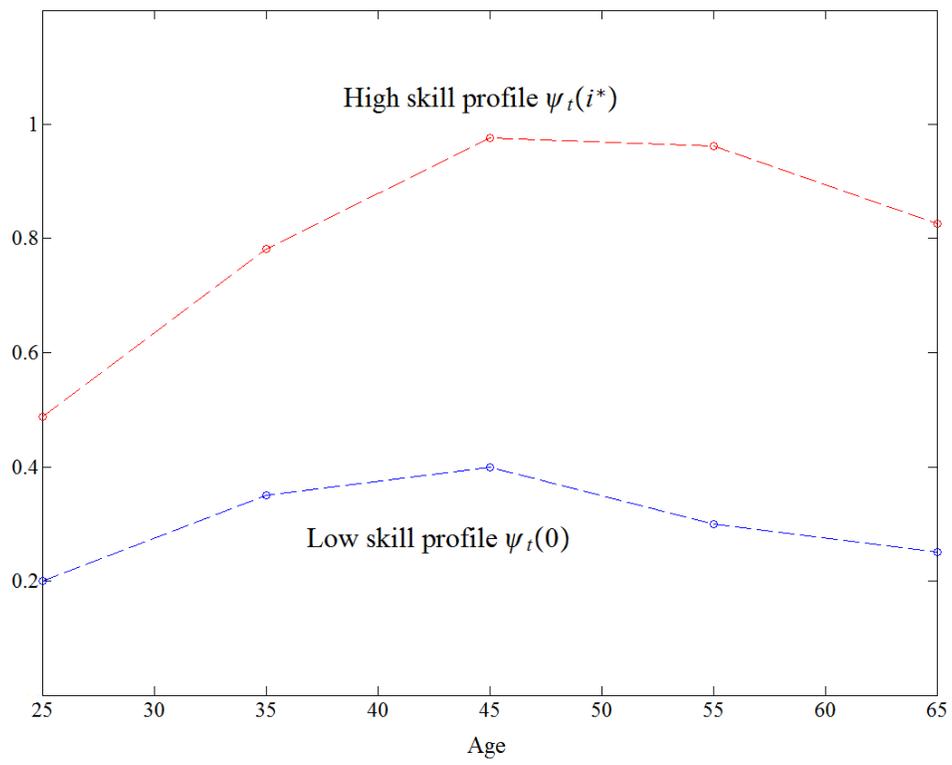
Figure 1: Skill Profiles

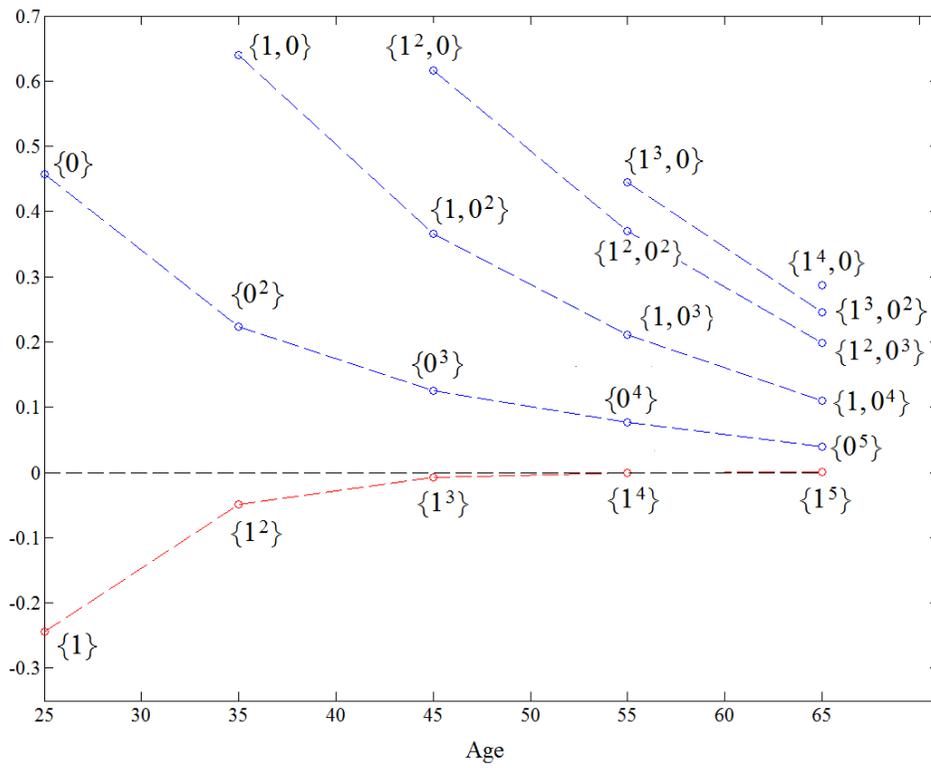Figure 2: Intratemporal Wedge across Types $\{\eta^t\}$

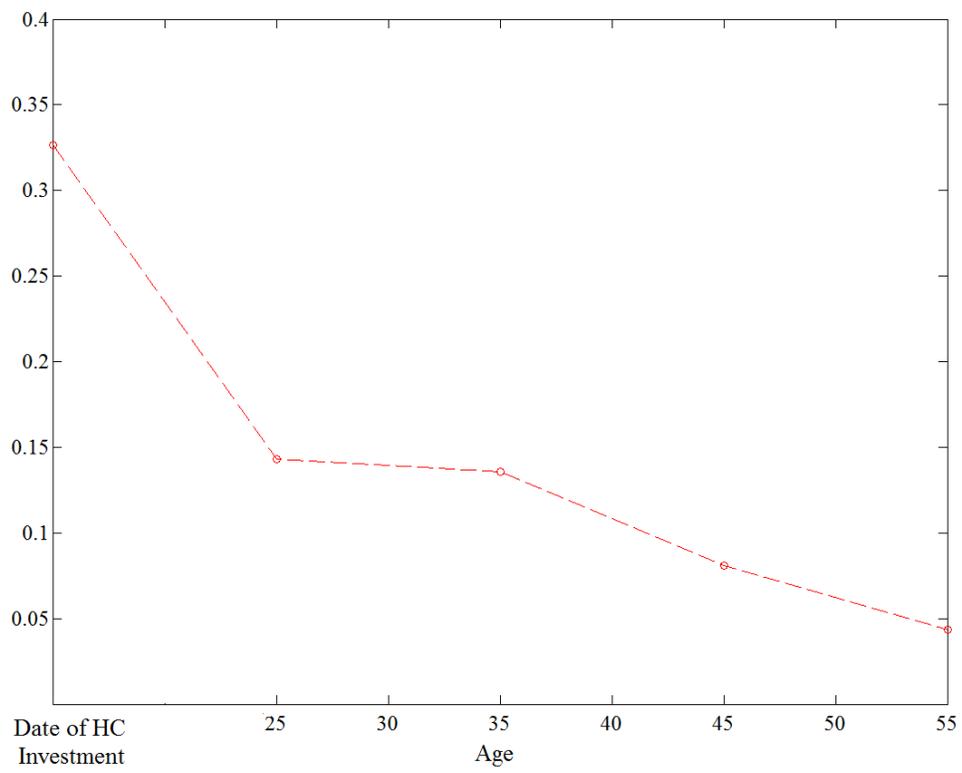Figure 3: Intertemporal Wedge for the High Skilled.

Figure 4: Optimal Contemporaneous and Deferred Capital Taxes across Types $\{\eta^t\}$