Working Paper Series

Notes on the Inflation Dynamics of the New Keynesian Phillips Curve

WP 07-04

Andreas Hornstein
Federal Reserve Bank of Richmond

THE FEDERAL RESERVE BANK OF RICHMOND
RICHMOND ▪ BALTIMORE ▪ CHARLOTTE

Notes on the Inflation Dynamics of the New Keynesian Phillips Curve*

Andreas Hornstein†

Federal Reserve Bank of Richmond Working Paper 2007-04

Abstract

These notes contain the derivations for results stated without proof in Hornstein (2007). First, I derive the log-linear approximation of the inflation dynamics in the Calvo-model with elements of backward-looking pricing when the approximation takes place around a positive average inflation rate. I derive a version of the "hybrid" New Keynesian Phillips Curve (NKPC) that can be estimated using standard GMM techniques. Second, I characterize the inflation dynamics implied by the NKPC when marginal cost follows an AR(1) process. For this purpose I derive the autocorrelation and crosscorrelation structure of inflation.

JEL Classification: C63, E31, E32

Keywords: Inflation, Nominal Rigidities, New Keynesian Phillips Curve

*Any opinions expressed in this paper are those of the author and do not necessarily reflect those of the Federal Reserve Bank of Richmond or the Federal Reserve System.

†Federal Reserve Bank of Richmond, `andreas.hornstein@rich.frb.org`

1. The NKPC at a steady state with positive inflation

We study a standard Calvo (1983)-type model with monopolistically competitive firms and nominal rigidities. There is a continuum of firms that produce differentiated products, that is, they face downward sloping demand curves. Firms set the nominal prices of their products, but they are limited in their ability to adjust their prices. In particular, whether a firm can optimally adjust its price is random, and the probability of price adjustment is constant over time. There is exogenous price indexation for firms that do not reoptimize their price. If a firm cannot readjust its price, the price increases in proportion to last period's aggregate inflation rate. This indexation scheme has been used by Christiano, Eichenbaum and Evans (2005). We derive a log-linear approximation of the equilibrium around a positive steady state inflation rate. The derivation follows Ascari (2004) and Cogley and Sbordone (2005, 2006). Finally, we show how the equilibrium conditions have to be modified when the price indexation scheme is replaced with "rule-of-thumb" price adjusters; that is, some firms never set their prices optimally. Rather they index their prices to the optimal price adjustment of the previous period, Galí and Gertler (1999).

1.1. The environment

Aggregate output is a Dixit-Stiglitz (1977) aggregator of a continuum of differentiated products on the unit interval

$$y_t = \left[\int_0^1 y_t\left(i\right)^{(\theta-1)/\theta} di \right]^{\theta/(\theta-1)}, \tag{1.1}$$

where the substitution elasticity is greater than one, $\theta > 1$. Each differentiated good, $y_t\left(i\right)$, is produced by a monopolistically competitive firm that sets the nominal price for its own product, $P_t\left(i\right)$. Assume that production of the final good is competitive, then the price index of the final good, P_t, is simply its unit cost given the prices for differentiated goods,

$$P_t \equiv \left[\int_0^1 P_t\left(j\right)^{1-\theta} dj \right]^{\frac{1}{1-\theta}}. \tag{1.2}$$

The demand function for a firm's differentiated product is declining in its relative price, $p_t\left(i\right) \equiv P_t\left(i\right)/P_t$,

$$y_t\left(i\right) = y_t p_t\left(i\right)^{-\theta}. \tag{1.3}$$

Production of the firm's differentiated product is assumed to be such that it yields a convex cost function

$$c_t\left[y_t\left(i\right)\right] = \frac{s_t y_t^{-\gamma}}{1+\gamma} \cdot y_t\left(i\right)^{1+\gamma}, \tag{1.4}$$

with constant own output elasticity, $1+\gamma$. The firm's own cost also depends on aggregate demand, and s_t will denote the aggregate marginal cost index in terms of the final good.

1.2. The price index

The evolution of the aggregate price index is determined by the optimal price setting of the monopolistically competitive firms and the limits on nominal price adjustment. In any period a firm has the opportunity to optimally reset its nominal price with probability $1-\alpha$. All firms that can adjust their price will choose the same price X_t since they are identical. For a firm that cannot reoptimize the nominal price, which happens with probability α, its nominal price is partially indexed to lagged general inflation, $\pi_{t-1} = P_{t-1}/P_{t-2}$, that is, its price will increase in proportion to last period's aggregate inflation rate

$$P_t(i) = \pi_{t-1}^{\rho} P_{t-1}(i), \tag{1.5}$$

with $\rho \in [0,1]$ being the indexation factor.

Substituting for the firms that adjust their prices and the firms whose prices are indexed to past inflation in the price index equation (1.2) we get

$$\begin{aligned} P_t &= \left\{(1-\alpha) X_t^{1-\theta} + \alpha \int_0^1 \left[\pi_{t-1}^{\rho} P_{t-1}(j)\right]^{1-\theta} dj \right\}^{\frac{1}{1-\theta}} \\ &= \left[(1-\alpha) X_t^{1-\theta} + \alpha \pi_{t-1}^{\rho(1-\theta)} \int_0^1 P_{t-1}(j)^{1-\theta} dj\right]^{\frac{1}{1-\theta}} \\ &= \left[(1-\alpha) X_t^{1-\theta} + \alpha \pi_{t-1}^{\rho(1-\theta)} P_{t-1}^{1-\theta}\right]^{\frac{1}{1-\theta}}. \end{aligned}$$

Dividing through by the aggregate price index we get an expression that relates current inflation to the optimal current relative price, $x_t \equiv X_t/P_t$, and current and past inflation,

$$1 = (1-\alpha) x_t^{1-\theta} + \alpha(\underbrace{\pi_{t-1}^{\rho} \pi_t^{-1}}_{\equiv \psi_t})^{1-\theta}. \tag{1.6}$$

1.3. Optimal price setting

A firm chooses its nominal price to maximize the expected present value of future profits in terms of the final good for the duration that it cannot reoptimize its price:

$$E_t \sum_{\tau=0}^{\infty} \alpha^{\tau} q_{t,\tau} \left\{p_{t+\tau}(i)\, y_{t+\tau}(i) - c_{t+\tau}\left[y_{t+\tau}(i)\right]\right\}, \tag{1.7}$$

where $q_{t,\tau}$ is the discount factor for period $t+\tau$ relative to period t. Given price indexation (1.5), the firm's relative price evolves according to

$$p_{t+\tau}(i) = \pi_{t+\tau}^{-1} \pi_{t+\tau-1}^{\rho} p_{t+\tau-1}(i) = \psi_{t+\tau} p_{t+\tau-1}(i) \tag{1.8}$$

during the time the firm cannot reoptimize its nominal price. Repeated substitution yields the τ-period ahead relative price as a function of the optimally chosen relative price and subsequent inflation

$$p_{t+\tau}(i) = \Pi_{j=1}^{\tau} \psi_{\tau+j} x_t(i) = \Psi_{t,\tau} x_t(i) \text{ for } \tau > 0, \tag{1.9}$$

and $\Psi_{t,0} \equiv 1$. Thus for a firm that sets its relative price in period t and does not have the opportunity to reset its price optimally in future periods, future demand can be written as a function of the optimally chosen relative price and future inflation

$$y_{t+\tau}(i) = y_{t+\tau} p_{t+\tau}(i)^{-\theta} = y_{t+\tau} \left[\Psi_{t,\tau} x_t(i)\right]^{-\theta}. \tag{1.10}$$

The FOC for the optimal relative price $x_t(i)$ is

$$\begin{aligned}
0 &= E_t \left[\sum_{\tau=0}^{\infty} \alpha^{\tau} q_{t,\tau} \left\{ \Psi_{t,\tau} y_{t+\tau}(i) \right.\right. \\
&\quad \left.\left. -\theta \left[\Psi_{t,\tau} x_t(i) - c'_{t+\tau}\left[y_{t+\tau}(i)\right]\right] y_{t+\tau} \Psi_{t,\tau}^{-\theta} x_t(i)^{-\theta-1} \right\} \right] \\
&= E_t \left[\sum_{\tau=0}^{\infty} \alpha^{\tau} q_{t,\tau} y_{t+\tau} \Psi_{t,\tau}^{1-\theta} \left\{ x_t(i) - \frac{\theta}{\theta-1} s_{t+\tau} \cdot \left[\Psi_{t,\tau} x_t(i)\right]^{-\theta\gamma} \Psi_{t,\tau}^{-1} \right\} \right] \\
&= E_t \left[\sum_{\tau=0}^{\infty} \alpha^{\tau} q_{t,\tau} y_{t+\tau} \Psi_{t,\tau}^{1-\theta} \left\{ x_t(i)^{1+\gamma\theta} - \frac{\theta}{\theta-1} s_{t+\tau} \Psi_{t,\tau}^{-(1+\theta\gamma)} \right\} \right].
\end{aligned} \tag{1.11}$$

We can solve expression (1.11) for the optimal relative price

$$x_t(i)^{1+\gamma\theta} = \frac{\theta}{\theta-1} \frac{C_t}{D_t} \text{ with} \tag{1.12}$$

$$C_t = E_t \left[\sum_{\tau=0}^{\infty} \alpha^{\tau} q_{t,\tau} g_{t,\tau} \Psi_{t,\tau}^{-(1+\gamma)\theta} s_{t+\tau} \right], \tag{1.13}$$

$$D_t = E_t \left[\sum_{\tau=0}^{\infty} \alpha^{\tau} q_{t,\tau} g_{t,\tau} \Psi_{t,\tau}^{1-\theta} \right], \tag{1.14}$$

where $g_{t,\tau} = y_{t+\tau}/y_t$ denotes the growth rate of aggregate demand from period t to period

$t + \tau$. Recursive definitions of C and D are

$$C_t = s_t + \alpha E_t \left[q_{t,1} g_{t,1} \psi_{t+1}^{-(1+\gamma)\theta} C_{t+1} \right] \tag{1.15}$$

$$D_t = 1 + \alpha E_t \left[q_{t,1} g_{t,1} \psi_{t+1}^{1-\theta} D_{t+1} \right]. \tag{1.16}$$

1.4. Log-linear approximation

We now derive a log-linear approximation of the price index equation (1.6) and the FOC for optimal price setting, (1.12), (1.15), and (1.16), at a steady state associated with some inflation rate $\bar{\pi}$, marginal cost $\bar{s}$, discount factor $\bar{q}$, and aggregate demand growth rate, $\bar{g}$. The steady state expressions for equations (1.6), (1.12), (1.15), and (1.16) are

$$1 = (1-\alpha)\,\bar{x}^{1-\theta} + \alpha \bar{\pi}^{(\rho-1)(1-\theta)}, \tag{1.17}$$

$$\bar{x}^{1+\theta\gamma} = \frac{\theta}{\theta-1}\frac{\bar{C}}{\bar{D}}, \tag{1.18}$$

$$\bar{C} = \frac{\bar{s}}{1-\alpha\bar{q}\bar{g}\bar{\pi}^{(1-\rho)(1+\gamma)\theta}}, \tag{1.19}$$

$$\bar{D} = \frac{1}{1-\alpha\bar{q}\bar{g}\bar{\pi}^{(1-\rho)(\theta-1)}}. \tag{1.20}$$

Note that the steady state values depend on the inflation rate.

The log-linear approximation of the price index equation (1.6) is

$$0 = (1-\alpha)\,\bar{x}^{1-\theta}\hat{x}_t + \alpha\bar{\pi}^{(\rho-1)(1-\theta)}\hat{\psi}_t.$$

Using the steady state condition (1.17) this can be rewritten as

$$\hat{x}_t = -\frac{\alpha\bar{\pi}^{(\rho-1)(1-\theta)}}{(1-\alpha)\,\bar{x}^{1-\theta}}\hat{\psi}_t = -\underbrace{\frac{\alpha\bar{\pi}^{(\rho-1)(1-\theta)}}{1-\alpha\bar{\pi}^{(\rho-1)(1-\theta)}}}_{\varphi_0}\hat{\psi}_t. \tag{1.21}$$

The log-linear approximations of the equations characterizing optimal prices setting are

$$\hat{x}_t = \frac{\hat{c}_t - \hat{d}_t}{1+\gamma\theta}, \tag{1.22}$$

$$\hat{c}_t = \underbrace{\left[1-\alpha\bar{q}\bar{g}\bar{\pi}^{(1-\rho)(1+\gamma)\theta}\right]}_{\varphi_3}\hat{s}_t + \underbrace{\left[\alpha\bar{q}\bar{g}\bar{\pi}^{(1-\rho)(1+\gamma)\theta}\right]}_{\varphi_2} E_t\left[\hat{z}_{t+1} - (1+\gamma)\,\theta\hat{\psi}_{t+1} + \hat{c}_{t+1}\right], \tag{1.23}$$

$$\hat{d}_t = \underbrace{\left[\alpha\bar{q}\bar{g}\pi^{(1-\rho)(\theta-1)}\right]}_{\varphi_1} E_t\left[\hat{z}_{t+1} - (\theta-1)\,\hat{\psi}_{t+1} + \hat{d}_{t+1}\right], \tag{1.24}$$

with $\hat{z}_{t+1} \equiv \hat{q}_{t,1} + \hat{g}_{t,1}$. Collecting terms, we can rewrite expressions (1.23) and (1.24) as

$$E_t\left[\left(1-\varphi_2 L^{-1}\right)\hat{c}_t + \varphi_2\left(1+\gamma\right)\theta\hat{\psi}_{t+1} - \varphi_2\hat{z}_{t+1}\right] = \varphi_3\hat{s}_t, \tag{1.25}$$

$$E_t\left[\left(1-\varphi_1 L^{-1}\right)\hat{d}_t + \varphi_1\left(\theta-1\right)\hat{\psi}_{t+1} - \varphi_1\hat{z}_{t+1}\right] = 0, \tag{1.26}$$

where L denotes the lag operators, $L^j x_t = x_{t-j}$ for all integers j.

Combining equations (1.21), (1.22), (1.25), and (1.26) defines the following difference equation in the change of the relative price of a firm that cannot reoptimize its nominal price

$$\begin{aligned} & -\varphi_0\left(1+\theta\gamma\right)E_t\left[\left(1-\varphi_1 L^{-1}\right)\left(1-\varphi_2 L^{-1}\right)\hat{\psi}_t\right] \\ = \; & E_t\left[\left(1-\varphi_1 L^{-1}\right)\left\{\varphi_3\hat{s}_t - \theta\left(1+\gamma\right)\varphi_2\hat{\psi}_{t+1}\right\}\right] \\ & + E_t\left[\left(\theta-1\right)\varphi_1\left(1-\varphi_2 L^{-1}\right)\hat{\psi}_{t+1} + \left(\varphi_2-\varphi_1\right)\hat{z}_{t+1}\right]. \end{aligned} \tag{1.27}$$

Collecting terms yields the following fourth-order difference equation in the inflation rate

$$E_t\left[\left(\mu_1+\mu_2 L^{-1}+\mu_3 L^{-2}\right)\left(1-\rho L\right)\hat{\pi}_t\right] = E_t\left[\left(\mu_4+\mu_5 L^{-1}\right)\hat{s}_t + \mu_6\hat{z}_{t+1}\right], \tag{1.28}$$

with

$$\begin{aligned} \mu_1 &= \left(1+\theta\gamma\right)\varphi_0, \\ \mu_2 &= \left(\theta-1\right)\varphi_1 - \left(1+\theta\gamma\right)\varphi_0\left(\varphi_1+\varphi_2\right) - \theta\left(1+\gamma\right)\varphi_2, \\ \mu_3 &= \left(1+\theta\gamma\right)\left(1+\varphi_0\right)\varphi_1\varphi_2, \\ \mu_4 &= \varphi_3, \\ \mu_5 &= -\varphi_1\varphi_3, \\ \mu_6 &= \varphi_2-\varphi_1. \end{aligned}$$

We can factor the polynomial in the lead operators on the LHS of equation (1.28) as

$$\begin{aligned} \left(\mu_1+\mu_2 L^{-1}+\mu_3 L^{-2}\right) &= \mu_3\left(\frac{\mu_1}{\mu_3}+\frac{\mu_2}{\mu_3}L^{-1}+L^{-2}\right) \\ &= \frac{\mu_3}{\lambda_1\lambda_2}\left(1-\lambda_1 L^{-1}\right)\left(1-\lambda_2 L^{-1}\right) \end{aligned}$$

where λ_i are the roots of the polynomial in the lead operator L^{-1}. Using the polynomial factorization on the LHS of equation (1.28), we get the NKPC

$$E_t\left[\left(1-\lambda_1 L^{-1}\right)\left(1-\lambda_2 L^{-1}\right)(1-\rho L)\hat{\pi}_t\right]=\kappa_1 E_t\left[\left(1+\frac{\mu_5}{\mu_4}L^{-1}\right)\hat{s}_t+\frac{\mu_6}{\mu_4}L^{-1}\hat{z}_t\right], \quad (1.29)$$

with $\kappa_1 \equiv \lambda_1\lambda_2\frac{\mu_4}{\mu_3}$. Equation (1.29) can be estimated using the same GMM methods as used in Galí and Gertler (1999) for the standard hybrid NKPC (1.31) below.

Conditional on the process for marginal cost, $\hat{s}_t$, and the effective discount factor, $\hat{z}_t$, and assuming that the roots of the polynomial in the lead operator are less than one in absolute value, one can solve the NKPC (1.29) forward and get inflation as a function of lagged inflation and of current and future marginal cost and the discount factor[1]

$$(1-\rho L)\hat{\pi}_t=\kappa_1 E_t\left[\left(1-\lambda_1 L^{-1}\right)^{-1}\left(1-\lambda_2 L^{-1}\right)^{-1}\left\{\left(1+\frac{\mu_5}{\mu_4}L^{-1}\right)\hat{s}_t+\frac{\mu_6}{\mu_4}L^{-1}\hat{z}_t\right\}\right]. \quad (1.30)$$

This expression allows us to interpret the two sources of inflation persistence. First, there is "extrinsic" persistence that inflation inherits through its dependence on the two driving forces, marginal cost and the discount factor. Since inflation is the expected present value of future marginal cost and discount rates, inflation will be more persistent the more persistent are its driving forces. Second, there is "intrinsic" persistence that is inherent to inflation through the backward-looking indexation scheme.

1.4.1. Approximation at a zero inflation rate

For an approximation around a steady state with zero inflation, $\bar{\pi}=1$, the coefficients on the lead terms in equation (1.27) are the same, $\varphi_1=\varphi_2$. Thus equation (1.27) simplifies to

$$\begin{aligned}-\varphi_0(1+\theta\gamma)\left(1-\varphi_1 L^{-1}\right)\hat{\psi}_t &= \varphi_3\hat{s}_t-\varphi_1\left[\theta(1+\gamma)-(\theta-1)\right]\hat{\psi}_{t+1}\\ -\varphi_0(1+\theta\gamma)\left[1-\varphi_1\left(1+\frac{1}{\varphi_0}\right)L^{-1}\right]\hat{\psi}_t &= \varphi_3\hat{s}_t.\end{aligned}$$

Substituting for the coefficients φ_0, φ_1, and φ_3 yields then the standard "hybrid" NKPC.

$$\left(1-\beta L^{-1}\right)(1-\rho L)\hat{\pi}_t=\underbrace{\left[\frac{1-\alpha}{\alpha}\frac{1-\alpha\beta}{1+\gamma\theta}\right]}_{\equiv\kappa_0}\hat{s}_t. \quad (1.31)$$

[1] A bounded solution for the inflation process may exist even if the roots λ_i are not all less than one in absolute value. In this case the roots characterizing the process for marginal cost and the discount factor have to be small enough, such that their product with the roots λ_i is less than one in absolute value.

1.5. Modifications with "rule-of-thumb" price adjusters

Gali and Gertler (1999) also assume that only a fraction $1-\alpha$ of all firms can adjust their price in any period. They do not allow for price indexation when firms cannot optimally reset their prices, $\rho = 0$. Instead Gali and Gertler (1999) introduce "rule-of-thumb" price adjusters that never set their nominal price optimally: rather they set their price, $X_{0,t}$, relative to an "average" price set in the last period, $X_{a,t-1}$, taking into account past inflation

$$X_{0,t} = X_{a,t-1}\frac{P_{t-1}}{P_{t-2}} = X_{a,t-1}\pi_{t-1}. \tag{1.32}$$

Let ω denote the fraction of these "rule-of-thumb" price setters. Then among all the firms that can adjust their nominal price, a fraction $1-\omega$ of producers will adjust their price optimally and a fraction ω will index their price to the last period's "average" new price. The "average" price set in the current period is then defined as

$$X_{a,t}^{1-\theta} = (1-\omega)\,X_t^{1-\theta} + \omega X_{0,t}^{1-\theta} \tag{1.33}$$

and the current period overall price index is

$$\begin{aligned} P_t^{1-\theta} &= \alpha\int_0^1 P_{t-1}^{1-\theta}(i)\,di + (1-\alpha)\left[(1-\omega)\,X_t^{1-\theta} + \omega X_{0,t}^{1-\theta}\right] \\ &= \alpha P_{t-1}^{1-\theta} + (1-\alpha)\,X_{a,t}^{1-\theta}. \end{aligned} \tag{1.34}$$

Dividing through equations (1.33) and (1.34) by the price level and using (1.32), we get

$$x_{a,t}^{1-\theta} = (1-\omega)\,x_t^{1-\theta} + \omega\left(\frac{\pi_{t-1}}{\pi_t}x_{a,t-1}\right)^{1-\theta}, \tag{1.35}$$

$$1 = \alpha\pi_t^{\theta-1} + (1-\alpha)\,x_{a,t}^{1-\theta}, \tag{1.36}$$

for the normalized average price adjustment $x_{a,t} \equiv X_{a,t}/P_t$ and the normalized optimal price adjustment x_t.

The steady state relations for the average price adjustment $\bar{x}_a$, optimal price adjustment, $\bar{x}$, and inflation, $\bar{\pi}$, are

$$\bar{x}_a = \bar{x}, \tag{1.37}$$

$$1 = \alpha\bar{\pi}^{\theta-1} + (1-\alpha)\,\bar{x}_a^{1-\theta}. \tag{1.38}$$

The log-linear approximations of (1.35) and (1.36) at the steady state are

$$\hat{x}_{a,t} = (1-\omega)\hat{x}_t + \omega\left(\hat{\pi}_{t-1} - \hat{\pi}_t + \hat{x}_{a,t-1}\right) \tag{1.39}$$
$$0 = (1-\alpha)\bar{x}_a^{1-\theta}\hat{x}_{a,t} - \alpha\bar{\pi}^{\theta-1}\hat{\pi}_t. \tag{1.40}$$

Using the steady state relations (1.37) and (1.38), we can combine equations (1.39) and (1.40), eliminate the average price adjustment, and get an expression in the inflation rate and the optimal price adjustment alone

$$\left[(1-\omega)\left(\bar{\pi}^{1-\theta} - \alpha\right)\right]\hat{x}_t = \left[\omega\bar{\pi}^{1-\theta} + (1-\omega)\alpha\right]\hat{\pi}_t - \left[\omega\bar{\pi}^{1-\theta}\right]\hat{\pi}_{t-1} \tag{1.41}$$

Equation (1.41) now replaces equation (1.21) in the case of inflation indexation, and we proceed with equations (1.22), (1.25), and (1.26) with no price indexation, $\rho = 0$.

2. Inflation dynamics of the NKPC

We now characterize persistence of inflation and its comovement with marginal cost when marginal cost follows a simple AR(1) process

$$\hat{s}_t = \delta\hat{s}_{t-1} + \varepsilon_t \tag{2.1}$$

with $0 < \delta < 1$ and ε_t is an iid shock with mean zero and variance σ_ε^2. For such an AR(1) process the second moments are

$$E\left[\hat{s}_t\hat{s}_t\right] = \frac{\sigma_\varepsilon^2}{1-\delta^2} = \sigma_s^2 \tag{2.2}$$
$$E\left[\hat{s}_t\hat{s}_{t-j}\right] = \delta^j\sigma_s^2$$

and the conditional expectations of marginal cost j periods ahead are

$$E_t\left[\hat{s}_{t+j}\right] = E_t\left[\delta\hat{s}_{t+j-1} + \varepsilon_{t+j}\right] = \delta E_t\left[\hat{s}_{t+j-1}\right] = \ldots = \delta^j\hat{s}_t. \tag{2.3}$$

2.1. Inflation dynamics of the simple NKPC

The basic NKPC approximates the inflation dynamics without indexation around a zero steady state inflation rate

$$E_t\left[\left(1-\beta L^{-1}\right)\pi_t\right] = \kappa_0\hat{s}_t + u_t. \tag{2.4}$$

We obtain the basic NKPC from equation (1.31) with $\rho = 0$. The shock, u_t, is simply added to the NKPC, and it is assumed to be iid with mean zero and variance σ_u^2, and uncorrelated with marginal cost.[2] We can solve (2.4) forward by repeatedly substituting for future inflation, and thereby obtain the current inflation rate as a discounted present value of future marginal cost

$$\hat{\pi}_t = \kappa_0 \sum_{j=0}^{\infty} \beta^j E_t \hat{s}_{t+j} + u_t. \tag{2.5}$$

Substituting for the expected future marginal cost from (2.3) we get

$$\hat{\pi}_t = \kappa_0 \sum_{j=0}^{\infty} \beta^j \delta^j \hat{s}_t + u_t = \frac{\kappa_0}{1 - \beta\delta} \hat{s}_t + u_t = a_0 \hat{s}_t + u_t. \tag{2.6}$$

Equation (2.6) is a reduced form relationship between current inflation and marginal cost. The relationship is reduced form since it incorporates the presumed equilibrium law of motion for marginal cost. If the law of motion for marginal cost changes, then the relation between inflation and marginal cost will change.

The second moments of the inflation rate process are given by

$$E\left[\hat{\pi}_t \hat{\pi}_{t-k}\right] = a_0^2 E\left[\hat{s}_t \hat{s}_{t-k}\right] + I_{[k=0]} \sigma_u^2 = \delta^k \left(a_0 \sigma_s\right)^2 + I_{[k=0]} \sigma_u^2 \tag{2.7}$$

where $I_{[.]}$ denotes the indicator function, $I_{[k=0]} = 1$ for $k = 0$ and zero otherwise. The expected cross-product of inflation and marginal cost is

$$E\left[\hat{\pi}_t \hat{s}_{t+k}\right] = a_0 E\left[\hat{s}_t \hat{s}_{t+k}\right] = \delta^k a_0 \sigma_s^2. \tag{2.8}$$

From the second moments we obtain the autocorrelation coefficients for inflation and the crosscorrelation coefficients of inflation and marginal cost as

$$Corr\left(\hat{\pi}_t, \hat{\pi}_{t-k}\right) = \delta^k \frac{a_0^2}{a_0^2 + \left(\sigma_u / \sigma_s\right)^2}, \tag{2.9}$$

$$Corr\left(\hat{\pi}_t, \hat{s}_{t+k}\right) = \delta^k \frac{a_0}{\left[a_0^2 + \left(\sigma_u / \sigma_s\right)^2\right]^{1/2}}. \tag{2.10}$$

In the simple NKPC the only source of inflation persistence is "extrinsic." Given the assumed law of motion for marginal cost, the inflation rate is positively correlated with marginal cost and inherits some of the persistence properties of marginal cost. In particular, the autocorrelation coefficients of inflation are simply scaled versions of the autocorrelation coefficients of

[2] When the inflation dynamics are approximated around a zero steady state inflation rate, one can interpret the shock u_t as a random disturbance to the demand elasticity parameter θ.

marginal cost. The scale factor depends on the importance of marginal cost for inflation, a_0, and the relative volatility of marginal cost shocks, σ_u/σ_s. The more important is marginal cost and the bigger is the relative volatility of marginal cost, the closer is the autocorrelation structure of inflation to that of marginal cost.

2.2. Inflation dynamics of the "hybrid" NKPC model

Consider now the "hybrid" NKPC with partial price indexation ρ

$$E_t\left[\left(1-\beta L^{-1}\right)(1-\rho L)\,\hat{\pi}_t\right]=\kappa_0\hat{s}_t+u_t. \tag{2.11}$$

Given the marginal cost process (2.1), the properties we have just derived for the inflation process of the simple NKPC now apply to the transformation of the inflation process,

$$\tilde{\pi}_t=(1-\rho L)\,\hat{\pi}_t=a_0\hat{s}_t+u_t. \tag{2.12}$$

The inflation rate itself is now an infinite sum of past transformed inflation rates $\tilde{\pi}$

$$\hat{\pi}_t=(1-\rho L)^{-1}\,\tilde{\pi}_t=\sum_{j=0}^{\infty}\rho^j\tilde{\pi}_{t-j}$$

The second moments of the inflation rate are now defined as follows:

$$\begin{aligned}
& E\left[\hat{\pi}_t\hat{\pi}_{t-k}\right]\\
= & \sum_{i\geq 0}\sum_{j\geq 0}\rho^i\rho^j E\left[\tilde{\pi}_{t-i}\tilde{\pi}_{t-k-j}\right]\\
= & \sigma_u^2\underbrace{\sum_{i\geq 0}\sum_{j\geq 0}\rho^i\rho^j I_{[i=k+j]}}_{A(k;\rho)}+(a_0\sigma_s)^2\underbrace{\sum_{i\geq 0}\sum_{j\geq 0}\rho^i\rho^j\delta^{|k+j-i|}}_{B(k;\rho,\delta)}.
\end{aligned}$$

For $k\geq 0$, we can write the two terms A and B as

$$\begin{aligned}
A(k;\rho) &= \rho^0\sum_{j\geq 0}\rho^j I_{[0=k+j]}+\rho^1\sum_{j\geq 0}\rho^j I_{[1=k+j]}+\rho^2\sum_{j\geq 0}\rho^j I_{[2=k+j]}+\ldots\\
&= \rho^k+\rho^{k+1}\rho+\rho^{k+2}\rho^2+\ldots\\
&= \rho^k\left(1+\rho^2+\rho^4+\rho^6+\ldots\right)\\
&= \rho^k\sum_{j\geq 0}\rho^{2j}=\frac{\rho^k}{1-\rho^2}
\end{aligned}$$

and

$$
\begin{aligned}
B(k;\rho,\delta) &= \sum_{i\geq 0}\sum_{j\geq 0}\rho^i\rho^j\delta^{|k+j-i|} \\
&= \sum_{i\geq 0}\rho^i\left\{\underbrace{\sum_{j=0}^{i-k-1}\rho^j\delta^{i-k-j}}_{B_{1i}(k)}+\underbrace{\sum_{j=\max\{0,i-k\}}^{\infty}\rho^j\delta^{k+j-i}}_{B_{2i}(k)}\right\} \\
B_{1i}(k;\rho,\delta) &= I_{[i>k]}\delta^{i-k}\sum_{j=0}^{i-k-1}(\rho/\delta)^j = I_{[i>k]}\delta^{i-k}\frac{1-(\rho/\delta)^{i-k}}{1-\rho/\delta} \\
B_{2i}(k;\rho,\delta) &= I_{[i\leq k]}\sum_{j=0}^{\infty}\rho^j\delta^{k+j-i}+I_{[i>k]}\sum_{j=i-k}^{\infty}\rho^j\delta^{j-(i-k)} \\
&= I_{[i\leq k]}\delta^{k-i}\sum_{j=0}^{\infty}(\rho\delta)^j+I_{[i>k]}\rho^{i-k}\sum_{j=i-k}^{\infty}(\rho\delta)^{j-(i-k)} \\
&= I_{[i\leq k]}\delta^{k-i}\frac{1}{1-\rho\delta}+I_{[i>k]}\rho^{i-k}\frac{1}{1-\rho\delta} \\
\sum_{i=0}^{\infty}\rho^iB_{1i}(k;\rho,\delta) &= \sum_{i=k+1}^{\infty}\rho^i\delta^{i-k}\frac{1-(\rho/\delta)^{i-k}}{1-\rho/\delta} \\
&= \frac{1}{1-\rho/\delta}\sum_{i=k+1}^{\infty}\left[\rho^k(\rho\delta)^{i-k}-\rho^k\rho^{2(i-k)}\right] \\
&= \frac{1}{1-\rho/\delta}\rho^k\left[\frac{\rho\delta}{1-\rho\delta}-\frac{\rho^2}{1-\rho^2}\right] \\
\sum_{i=0}^{\infty}\rho^iB_{2i}(k;\rho,\delta) &= \left[\sum_{i=0}^{k}\rho^i\delta^{k-i}+\sum_{i=k+1}^{\infty}\rho^{2i-k}\right]\frac{1}{1-\rho\delta} \\
&= \left[\delta^k\frac{1-(\rho/\delta)^{k+1}}{1-\rho/\delta}+\rho^k\frac{\rho^2}{1-\rho^2}\right]\frac{1}{1-\rho\delta} \\
B(k;\rho,\delta) &= \rho^k\left[\frac{\rho\delta(1-\rho^2)-\rho^2(1-\rho\delta)+\rho^2(1-\rho/\delta)-(\rho/\delta)(1-\rho^2)}{(1-\rho/\delta)(1-\rho\delta)(1-\rho^2)}\right] \\
&\quad +\delta^k\frac{1}{(1-\rho/\delta)(1-\rho\delta)} \\
&= \left[\rho^{k+1}\frac{\delta-1/\delta}{1-\rho^2}+\delta^k\right]\frac{1}{(1-\rho/\delta)(1-\rho\delta)} \\
&= \left[\rho^{k+1}\frac{1}{\delta}\frac{\delta^2-1}{1-\rho^2}+\delta^k\right]\frac{1}{(1-\rho/\delta)(1-\rho\delta)} \\
&= \left[\delta^k-\frac{\rho}{\delta}\frac{1-\delta^2}{1-\rho^2}\rho^k\right]\frac{1}{(1-\rho/\delta)(1-\rho\delta)}
\end{aligned}
$$

The crossproducts of inflation and marginal cost are

$$\begin{aligned} E\left[\hat{\pi}_t \hat{s}_{t+k}\right] &= E\left[(1-\rho L)\left(a_0 \hat{s}_t + u_t\right)\hat{s}_{t+k}\right] \\ &= \sum_{j=0}^{\infty} \rho^j a_0 E\left[\hat{s}_{t-j}\hat{s}_{t+k}\right] \\ &= a_0 \sigma_s^2 \sum_{j=0}^{\infty} \rho^j \delta^{|k+j|} = a_0 \sigma_s^2 C\left(k;\rho,\delta\right). \end{aligned}$$

If $k \geq 0$ then

$$C\left(k;\rho,\delta\right) = \delta^k \sum_{j=0}^{\infty} \rho^j \delta^j = \frac{\delta^k}{1-\rho\delta}, \tag{2.13}$$

and if $k < 0$ then

$$\begin{aligned} C\left(k;\rho,\delta\right) &= \sum_{j=0}^{-k-1} \rho^j \delta^{-(k+j)} + \sum_{j=-k}^{\infty} \rho^j \delta^{(k+j)} \\ &= \delta^{-k} \frac{1-\left(\rho/\delta\right)^{-k}}{1-\rho/\delta} + \rho^{-k} \frac{1}{1-\rho\delta} \\ &= \frac{1}{1-\rho/\delta}\left\{\delta^{-k} - \rho^{-k}\frac{\rho}{\delta}\frac{1-\delta^2}{1-\rho\delta}\right\}. \end{aligned} \tag{2.14}$$

We can now calculate the autocorrelation coefficients for the inflation rate and the cross-correlations of inflation and marginal cost as

$$Corr\left(\hat{\pi}_t, \hat{\pi}_{t-k}\right) = \frac{\left(\sigma_u/\sigma_s\right)^2 A\left(k;\rho\right) + a_0^2 B\left(k;\rho,\delta\right)}{\left(\sigma_u/\sigma_s\right)^2 A\left(0;\rho\right) + a_0^2 B\left(0;\rho,\delta\right)} \tag{2.15}$$

$$Corr\left(\hat{\pi}_t, \hat{s}_{t+k}\right) = \frac{a_0 C\left(k;\rho,\delta\right)}{\left[\left(\sigma_u/\sigma_s\right)^2 A\left(0;\rho\right) + a_0^2 B\left(0;\rho,\delta\right)\right]^{1/2}}. \tag{2.16}$$

In the "hybrid" NKPC there are two sources of inflation persistence, "intrinsic" and "extrinsic" persistence. Equations (2.15) and (2.16) illustrate how the two sources of inflation persistence ultimately affect overall inflation persistence. We can see that persistence of marginal cost, a high δ, is relatively more important for the dynamics of inflation if marginal cost is quantitatively important in the reduced form NKPC, that is, the coefficient a_0 is larger, and if the shocks to the NKPC are small relative to the volatility of marginal cost. Alternatively, if marginal cost is not important or if shocks to the NKPC are relatively large, the inflation persistence can only arise through the inherent persistence of inflation, that is, a high ρ.

2.3. Inflation dynamics at positive steady state inflation

In the case of an AR(1) process for marginal cost, it is straightforward to derive the persistence properties of inflation around a positive steady state inflation rate. Adding a shock u_t to the modified NKPC and assuming that $E_t[\hat{z}_{t+1}] = 0$ equation (1.29) simplifies to

$$E_t\left[\left(1-\lambda_1 L^{-1}\right)\left(1-\lambda_2 L^{-1}\right)(1-\rho L)\,\hat{\pi}_t\right] = \kappa_1\left(1+\frac{\mu_5}{\mu_4}\delta\right)\hat{s}_t + u_t. \tag{2.17}$$

Assuming that the roots of the polynomial and the persistence of marginal cost are such that $|\delta\lambda_i| < 1$, we can divide through by the inverse lead polynomials and get

$$\begin{aligned} E_t\left[(1-\rho L)\,\hat{\pi}_t\right] &= \kappa_1\left(1+\delta\frac{\mu_5}{\mu_4}\right)E_t\left[\left(1-\lambda_1 L^{-1}\right)^{-1}\left(1-\lambda_2 L^{-1}\right)^{-1}\hat{s}_t\right] + u_t \\ &= \underbrace{\kappa_1\frac{1+\delta\mu_5/\mu_4}{(1-\lambda_1\delta)(1-\lambda_2\delta)}}_{\equiv a_1}\hat{s}_t + u_t, \end{aligned} \tag{2.18}$$

which is formally equivalent to equation (2.12) for the "hybrid" NKPC, but now the coefficient a_1 is a function of the average inflation rate and the demand elasticity θ.

References

[1] Ascari, Guido. 2004. "Staggered Prices and Trend Inflation: Some Nuisances." *Review of Economic Dynamics* 7 (3), 642-667.

[2] Calvo, Guillermo A. 1983. "Staggered Prices in a Utility Maximizing Framework." *Journal of Monetary Economics* 12 (3), 383-398.

[3] Christiano, Lawrence J., Martin Eichenbaum, and Charles L. Evans. 2005. "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy." *Journal of Political Economy* 113 (1), 1-45.

[4] Cogley, Timothy and Argia M. Sbordone. 2005. "A Search for a Structural Phillips Curve." Federal Reserve Bank of New York Staff Report 203.

[5] Cogley, Timothy and Argia M. Sbordone. 2006. "Trend Inflation and Inflation Persistence in the New Keynesian Phillips Curve." Federal Reserve Bank of New York Staff Report 270.

[6] Dixit, Avinash K. and Joseph E. Stiglitz. 1977. "Monopolistic Competition and Optimum Product Diversity." *American Economic Review* 67 (3), 297-308.

[7] Galí, Jordi and Mark Gertler. 1999. "Inflation Dynamics: A Structural Econometric Analysis." *Journal of Monetary Economics* 44 (2), 195-222.

[8] Hornstein, Andreas. 2007. "Evolving Inflation Dynamics and the New Keynesian Phillips Curve." Federal Reserve Bank of Richmond *Economic Quarterly*, forthcoming.