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Market-based Incentives

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Abstract

In this paper, we study market-induced, external incentives similar to career concerns jointly with standard, contractual incentives linking compensation to performance. We consider a dynamic principal-agent problem in which the agent's outside option is determined endogenously in a competitive labor market. In equilibrium, strong performance increases the agent's market value. When this value becomes sufficiently high, the threat of the agent quitting forces the principal to increase the agent's compensation. The prospect of obtaining this raise gives the agent an incentive to exert effort, which reduces the need for standard incentives. In fact, whenever the agent's option to quit is sufficiently close to being "in the money," the market-induced incentive eliminates the need for standard incentives altogether: compensation becomes completely insensitive to current performance.

Keywords: incentives, long-term contracts, career concerns, moral hazard, limited commitment

JEL codes: D82, D86, J33

1 Introduction

Misaligned incentives can induce wasteful individual behavior and lead to disastrous aggregate outcomes.[1] For that reason, to understand how incentives are provided in the economy is one of the central questions in economics.

Two main sources of incentives have been identified in the literature: direct incentives specified explicitly in contracts between the counterparties to a given economic relationship, and external incentives coming from outside of the relationship. These two sources of incentives,

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[1] To give just one recent example, the Federal Reserve (2011) states that "Risk-taking incentives provided by incentive compensation arrangements in the financial services industry were a contributing factor to the financial crisis that began in 2007."

however, have been studied almost exclusively in separation from one another. The traditional principal-agent literature studies direct incentives obtained by contractually connecting the agent's compensation to her performance. But this literature does not consider the impact that the agent's performance may have on her outside options.[2] In polar contrast, the literature on career concerns studies the external incentives stemming from the impact the agent's performance has on her reputation and hence on wages she can earn in the future. But it ignores contractual provision of incentives by assuming that spot wages are paid every period.[3] In reality, both sources of incentives are present in many, if not most, long-term economic relationships.[4] It is therefore important to study direct and external incentives together and examine how they jointly affect behavior.

In this paper, we study the interaction between direct and external incentives. Our model allows for fully flexible, long-term contracts and also captures the agent's concern for the value she can obtain outside the present relationship. We obtain a novel characterization of the optimal mix of incentives: external incentives are fully sufficient when the agent is a new hire or has produced a record of strong performance; contractual incentives become needed only after an extended period of weak performance.

There are two frictions in our model: moral hazard and limited commitment. The agent /worker's effort, which is unobservable to the principal/firm, determines the drift of the worker's productivity process. The output the worker produces for the firm is proportional to her productivity. The worker's effort, therefore, has a persistent impact on both the revenue of the current firm and on the worker's own productivity in the future. The worker cannot commit to staying with the firm forever: at any time, she can quit and rejoin the labor market, where she can get a new contract with a new firm. The value the worker can obtain under the new contract—her market value—is increasing in her own productivity. Even if the compensation she receives from her current employer does not directly reward her current performance, the worker may want to exert effort just to improve her future productivity and enhance her market value. We will refer to this external motive for exerting effort as the market-based incentive.[5]

The strength of the market-based incentive depends on the worker's distance to quitting. In particular, this incentive is strongest at the worker's quitting boundary, where the worker's market value is equal to the continuation value she draws from the long-term contract with her current firm. The reason for it is that when the worker is about to quit, the firm cannot provide insurance.

[2] See, e.g., Rogerson (1985), Phelan and Townsend (1991), and Sannikov (2008).

[3] See Fama (1980), Holmstrom (1982), and the large body of literature that has followed.

[4] As an example, consider the CEO of a publicly traded corporation. Typically, her compensation will be contractually connected to the corporation's performance via grants of equity or options, which gives her a direct incentive to create shareholder value. But the company's performance also affects the CEO's reputation and, thus, her standing in the broader market for business executives. This indirect exposure to the corporation's performance is a second channel through which the CEO is incentivized to create shareholder value.

[5] This incentive is similar to career concerns, but there are important differences between the two, which we discuss in section 1.1.

Indeed, away from the quitting boundary, the long-term contract (at least partially) insures the worker against her idiosyncratic productivity shocks: the continuation value inside the contract is less sensitive to the worker's observed performance than the worker's outside market value. Insurance implies that when the worker receives positive productivity shocks and her performance improves, her continuation value inside the contract raises more slowly than does her outside market value, which brings the worker closer to her quitting boundary. When the quitting boundary is reached, if performance continues to improve, the continuation value inside the contract no longer can raise more slowly than the worker's outside market value. If it did, it would drop below the market value, and the worker would quit.[6] At the boundary, therefore, the contract does not provide any insurance to the worker, i.e., the worker's continuation value is highly sensitive to her performance. Since performance is the sum of shocks and effort, the continuation value is at this point highly sensitive to the worker's effort, which means the worker's incentive to exert effort is strong. This incentive is market based, as it is brought about by the increase in the worker's market value. Therefore, the market-based incentive is always strong at the quitting boundary.

When the worker is not at the quitting boundary but close to it, the long-term contract can provide some insurance to the worker but not much, because the quitting boundary is likely to be reached again soon. The sensitivity to the worker's performance imposed by the possibility of quitting therefore remains high, i.e., the market-based incentive remains strong near the quitting boundary. The farther away from the quitting boundary the worker is, the lower the risk of quitting, the more fully the long-term contract can insure the worker, and, thus, the weaker the market-based incentive becomes.

We show that the optimal long-term contract relies exclusively on the market-based incentive when the distance to the quitting boundary is below a threshold. Above this threshold, the market-based incentive has to be supplemented with standard performance compensation. The optimal contract, therefore, has two phases: a rigid-wage phase and a pay-for-performance phase.

The contract starts out in the rigid-wage phase. New hires are made at the quitting boundary (i.e., the worker gets from the firm no more than her market value), where the market-based incentive is strong enough to by itself elicit optimal effort from the worker. Compensation paid to the worker is downward-rigid, as in Harris and Holmstrom (1982): it is increased only as needed to match the worker's outside option and keep her from quitting, otherwise compensation is constant. Strong performance induces frequent raises in compensation. Weak performance means no raise, but compensation never decreases in this phase of the contract. In this phase, thus, compensation is back-loaded, i.e., expected to increase over time, and is not volatile: compensation does not respond to the worker's performance outside of instances in which the

[6] As in Harris and Holmstrom (1982), there is no economic role for job transitions in our homogeneous-firm model of the labor market. We thus derive the optimal long-term contract under the assumption that workers do not quit if indifferent. The alternative assumption leads to the exact same equilibrium processes for effort and compensation.

worker's quitting constraint binds.

The contract enters the pay-for-performance phase if the worker experiences a sufficiently long streak of poor performance and her market value drops significantly below the contract continuation value. At this point, the market-based incentive becomes weak, i.e., insufficient to by itself elicit worker effort. Consequently, compensation is no longer downward-rigid and becomes similar to the optimal compensation from the pure moral hazard model, as in, e.g., Sannikov (2008): to provide sufficient incentives, compensation must be sensitive to contemporaneous performance—both on the upside and on the downside. As well, compensation in this phase of the contract is front-loaded, i.e., expected to decrease over time.

In which phase the contract spends more time depends on the parameters of the model, and in particular on the expected growth of worker productivity. If productivity tends to grow over time, the worker's market value tends to increase, the quitting constraint binds often, which makes market-based incentives strong frequently and performance pay needed rarely. With a sufficiently large positive trend in worker productivity, the probability that contract-based incentives are ever used can be arbitrarily small.

As an extension of our model, we study the possibility that not only workers but also firms lack commitment. In particular, we follow Phelan (1995) in assuming that firms can fire workers upon incurring a firing cost. In this extension, thus, in addition to the worker's quitting constraint, we have a firm's participation, or firing, constraint. We show that if the firing cost is not too large, the worker is always exposed to her market value risk and, thus, market-based incentives are always strong. In this extension, optimal compensation has a "sticky wage" structure: small positive or negative performance shocks do not affect compensation; large positive (negative) shocks increase (decrease) the compensation paid to the worker.

In order to characterize the solution to our model analytically, we make several assumptions commonly used in the dynamic contracting literature. We consider a binary effort choice set and impose a sufficient condition on the parameters of the model (Assumption 1) for high effort to be optimal at all times. Constant absolute risk aversion (CARA) preferences and Gaussian shocks let us reduce to one the dimension of the state space sufficient for a recursive representation of the contracting problem. The optimal contract is then characterized by solving an ordinary differential equation.[7] Although needed for analytical tractability, these assumptions are not necessary for the existence of market-based incentives. Essential for the existence of market-based incentives are workers' limited commitment and a positive impact of workers' on-the-job effort on their market value. These conditions seem very plausible. The latter condition, in particular, is similar to learning-by-doing. It will be satisfied whenever putting in effort on the job helps a worker acquire any kind of skill or experience that is valued in the labor market.

Our model provides several testable predictions, which we discuss in Section 7. In particular,

[7] Because the lower bound is a reflecting rather than an absorbing barrier for the state variable in our model, the differential equation characterizing the equilibrium contract does not satisfy standard regularity conditions for the existence of a solution. We develop a change of variable technique to solve this problem. This technique can be useful in studying other contracting problems with reflective barrier dynamics.

having both a rigid-wage phase and a pay-for-performance phase, the equilibrium contract from our model can generate wage change frequencies consistent with empirical evidence.

1.1 Relation to the literature

Our paper is related to the literature on career concerns, e.g., Holmstrom (1982), Fudenberg and Tirole (1986). In the career concerns model, the worker's market value increases with her effort because of the market's slow learning about the worker's persistent ability/productivity level. By exerting effort, the worker can manipulate the market's belief and increase the spot wage she receives. In our model, the market's belief about the worker's productivity is trivially correct at all times, as the worker's productivity is public information. Because our model does not have complicated learning dynamics, we are able to solve it allowing for fully flexible, long-term compensation contracts, i.e., without assuming that spot wages are paid every period. The worker cares about her market value not because she can manipulate the market's belief but because her market value provides a floor under the continuation value she receives in the optimal long-term contract.[8]

Gibbons and Murphy (1992) study contract- and market-induced incentives in the learning environment of Holmstrom (1982). They restrict attention to one-period, linear compensation contracts. This restriction makes their model tractable because, after any deviation, the deviating agent's optimal effort strategy coincides with the equilibrium strategy. Our model, in contrast, is tractable without exogenous contract restrictions.

We build on the extensive literature on optimal long-term contracts in environments in which private information induces a trade-off between incentives and insurance, e.g., Townsend (1982), Rogerson (1985), Spear and Srivastava (1987), Phelan and Townsend (1991). In particular, we follow Sannikov (2008) in studying dynamic moral hazard in continuous time.[9] Because this literature takes the agent's outside option parametrically, it does not capture external, market-based incentives. Our paper shows that external incentives arise in the dynamic principal-agent problem if a) strong performance enhances the agent's outside value, and b) the agent cannot contractually commit to not leaving her current relationship should quitting become beneficial. We show that external incentives change significantly the structure of the optimal contract. In particular, in addition to the pay-for-performance phase familiar from dynamic moral hazard models, with external incentives the contract also has a rigid-wage phase in our model.

Our paper also builds on the literature on optimal contracting with commitment frictions. In particular, we add a dynamic moral hazard problem to a version of the Harris and Holmstrom (1982) model of the labor market with risk-neutral firms/principals competing for risk-averse

[8]He et al. (2013) and Prat and Jovanovic (2014) study optimal long-term contracts with learning under full commitment, i.e., without external incentives. In Williams (2015), the persistent private information problem arises out of the agent's access to hidden savings.

[9]Levin (2003), DeMarzo and Sannikov (2006), and Garrett and Pavan (2012) are recent examples of studies on the provision of incentives to risk-neutral agents. These papers belong to a distinct literature in which the trade-off between incentive and insurance, central to our analysis, is absent.

workers/agents who seek insurance against persistent idiosyncratic productivity shocks. Firms are able to commit to long-term contracts, but workers cannot.[10] As in Harris and Holmstrom (1982), one-sided commitment leads to downward-rigid compensation in our model, but only when the worker is close enough to quitting. Different from Harris and Holmstrom (1982), due to moral hazard, downward-rigidity of compensation does not hold when the quitting constraint is sufficiently slack.[11]

There exist a small number of studies that, like we do here, examine optimal contracts under the two frictions of private information and limited commitment. Two studies closely related to our paper are Thomas and Worrall (1990, Section 8) and Phelan (1995). In these papers, however, external incentives do not arise because the agent's outside option does not depend on her past performance. In Atkeson (1991), the outside option of the agent (a borrowing country) does depend on her actions (investment). For this reason, although that paper asks a different question, we expect that market-based incentives exist in that environment. These incentives are probably weak in that model because persistence in the impact of the private action (investment) on the value of the outside option (autarky) is not very strong. In our model, effort has a permanent effect on the worker's outside option, which makes market-based incentives much stronger and easier to identify.[12]

Bohren (2013) studies Perfect Public Equilibria in a class of stochastic games between a long-run player and a sequence of short-run players, where the long-run player's actions have a persistent impact on future payoffs via a publicly observable state variable. Persistence generates incentives for the long-run player and allows for equilibria with payoffs dominating the static best response. These incentives are similar to the market-based incentives we study in this paper in a dynamic principal-agent model.

Organization The model environment is formally defined in Section 2. Sections 3 and 4 study single-friction versions of our model, with full commitment in Section 3 and full information in Section 4. Optimal contracts from these models serve as benchmarks that we use to solve the full model in Section 5. Section 6 considers two extensions: log utility and geometric Brownian productivity shocks, and limited commitment on the firm side. Section 7 discusses testable predictions of the model. Proofs of all results formally stated in the text are relegated to the Appendix.

[10] A similar environment with one-sided commitment, but without moral hazard, is used in Krueger and Uhlig (2006) to study equilibrium with long-term financial insurance contracts.

[11] Lamadon (2014) shows a different mechanism by which wage reductions can be part of an optimal long-term contract between a risk-neutral, committed firm and a risk-averse, uncommitted worker. In his model, match-specific productivity shocks are transmitted into wages, so the wage declines when match quality deteriorates.

[12] Ales et al. (2014) study a political-economy model with policymaker private information and lack of commitment.

2 A labor market with long-term contracts

2.1 Informal description

We consider a labor market populated with a large number of workers and a potentially larger number of firms operating under free entry. For concreteness, we will assume that one firm hires one worker.[13] Time t runs from zero to infinity. At $t = 0$, each worker is characterized by her initial productivity level y_0. Each worker costlessly and instantaneously matches with one of the continuum of homogeneous firms. The firm offers the worker a long-term contract. If the worker rejects it, she can costlessly and instantaneously get a match with another firm out of the continuum. The firm knows the worker's initial productivity y_0, the stochastic process governing the evolution of the worker's productivity in the future (described below), and the worker's outside options. In particular, the firm knows that the worker's cost of getting a new match instantaneously at $t = 0$ is zero. Therefore, the firm offers a long-term contract that maximizes the worker's lifetime utility subject to the firm's break-even condition, and the worker accepts this contract. The value of lifetime utility delivered to the worker this way is referred to as the worker's market value, and is denoted by $V(y_0)$. The function V is determined in equilibrium.

As the firm designs the long-term contract maximizing the worker's utility, it faces two frictions: moral hazard and the worker's lack of commitment to the contract. The moral hazard friction is a standard shirking problem with an unobservable action the worker takes at each t. The commitment friction results from the worker's right to quit the current firm and reenter the labor market at any date t. The structure of the labor market at $t > 0$ is the same as it is at $t = 0$. The worker can costlessly and instantaneously get a match with a new firm that will offer her a new long-term contract commencing at t and delivering to the worker her market value $V(y_t)$, where y_t is the worker's productivity at time t. As the new firm does not make the worker any more productive, there is no reason for quits to actually happen in equilibrium. Despite that, the possibility of quitting restricts the set of contracts that the firm can offer to the worker at $t = 0$. In particular, the worker's time-t value of continuing with the original firm, denoted by W_t, can never be smaller than her market value $V(y_t)$. We will refer to $W_t \geq V(y_t)$ as the quitting constraint. Whenever $W_t = V(y_t)$, we will say that the quitting constraint binds, or that the worker has reached her quitting boundary.

The worker's market value function V is an equilibrium object determined by a fixed-point condition: if firms perceive V as workers' outside option, then the values that firms actually deliver to the workers must be consistent with V. If, for example, the firms perceive V to be very high, they expect the quitting constraint to bind often, which restricts their ability to provide insurance and makes the actual value they deliver to workers smaller than V. This cannot be an equilibrium. The equilibrium V is determined at the point at which the perceived

[13] As long as each worker's performance is observable, our results would be unchanged if firms in the model hired multiple workers.

and the actual values are the same.

2.2 Formal model

Workers' productivity y_t follows a Brownian motion with drift. Specifically, let $\boldsymbol{z}$ be a standard Brownian motion $\boldsymbol{z} = \{z_t, \mathcal{F}_t; t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. A worker's productivity process $\boldsymbol{y} = \{y_t; t \geq 0\}$ is $y_0 \in \mathbb{R}$ at $t = 0$ and evolves according to

$$dy_t = a_t dt + \sigma dz_t. \tag{1}$$

The drift in a worker's productivity at t, a_t, is privately controlled by the worker via a costly action $a_t \in \{a_l, a_h\}$ with $a_l < a_h$. The constant $\sigma > 0$ is independent of the worker's action.

Workers are heterogeneous in the initial level of their productivity y_0, in the realized paths of their productivity shocks $\{z_t; t > 0\}$, and, potentially, in the action path $\{a_t; t \geq 0\}$ they choose. The action path $\{a_t; t \geq 0\}$ taken by each worker is her private information.[14] The structure of the productivity process and each worker's productivity level y_t are public information at all times. In particular, y_t is observable not only to the firm for which the worker works at t, but also to all other firms.

We adopt a simple production function in which the revenue the worker generates for the firm equals the worker's productivity y_t at all times during her employment with the firm. In a long-term employment contract, the firm collects revenue $\{y_t; t \geq 0\}$ and pays compensation $\{c_t; t \geq 0\}$ to the worker. We will identify compensation c_t with the worker's consumption at all $t \geq 0$.[15] Formally, a long-term contract a firm and a worker enter at $t = 0$ specifies an action process $\boldsymbol{a} = \{a_t; t \geq 0\}$ for the worker to take, and a compensation/consumption process $\boldsymbol{c} = \{c_t; t \geq 0\}$ the worker receives. Processes $\boldsymbol{a}$ and $\boldsymbol{c}$ must be adapted to the information available to the firm.

We assume that firms and workers discount future payoffs at a common rate r. The firm's expected profit from a contract $(\boldsymbol{a}, \boldsymbol{c})$ is given by

$$\mathbb{E}^{\boldsymbol{a}} \left[\int_0^\infty r e^{-rt} (y_t - c_t) dt \right],$$

where $\mathbb{E}^{\boldsymbol{a}}$ is the expectation operator under the action plan $\boldsymbol{a}$.

All workers have identical preferences over compensation/consumption processes $\boldsymbol{c}$ and action processes $\boldsymbol{a}$. These preferences are represented by the expected utility function

$$\mathbb{E}^{\boldsymbol{a}} \left[\int_0^\infty r e^{-rt} U(c_t, a_t) dt \right].$$

[14]This form of private information is often referred to as moral hazard.

[15]We can think of the worker's savings or financial wealth as being observable and thus contractually controlled by the firm.

To make our model tractable analytically, we abstract from wealth effects in the provision of incentives. That is, we assume constant absolute risk aversion (CARA) with respect to consumption by taking

$$U(c_t, a_t) = u(c_t)\phi^{1_{a_t=a_l}},$$

where $u(c_t) = -\exp(-c_t) < 0$, $0 < \phi < 1$, and $1_{a_t=a_l}$ is the indicator of the low-effort action a_l at time t. The high-effort action a_h is costly to the worker in current utility terms because $U(c, a_h) = u(c) < u(c)\phi = U(c, a_l)$ for all c.[16] In Section 6, we discuss the extent to which our results depend on this form of the utility function.

Firms can commit to long-term contracts, but workers cannot. Contracts therefore must satisfy worker's participation (or quitting) constraints. For a worker with initial productivity $y_0 \in \mathbb{R}$, a contract $(\boldsymbol{a}, \boldsymbol{c})$ induces a continuation value process $\boldsymbol{W} = \{W_t; t \geq 0\}$ given by

$$W_t = \mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rs}U(c_{t+s}, a_{t+s})ds\,|\mathcal{F}_t\right]. \tag{2}$$

Contract $(\boldsymbol{a}, \boldsymbol{c})$ satisfies the worker's quitting constraints if at all dates and states

$$W_t \geq V(y_t), \tag{3}$$

where $V(y_t)$ is the value a worker with productivity y_t can obtain if she quits and rejoins the labor market. This market value is determined in equilibrium. We show later (in Proposition 1) that V is strictly increasing.[17]

The quitting constraint (3) is standard in models of optimal contracts with limited commitment (e.g., Thomas and Worrall (1988)). It also resembles the lower-bound constraint on the continuation value W_t used in many principal-agent models with private information (e.g., Atkeson and Lucas (1995) and Sannikov (2008)). A notable difference here, however, is that the lower bound in those models is given by some fixed value, whereas in (3) the lower bound $V(y_t)$ fluctuates with the worker's productivity y_t. As we show later, this difference has important implications for the provision of incentives to the worker at the quitting boundary.

In this paper, we adopt the convention that when the quitting constraint (3) binds, i.e., when the worker is indifferent to quitting, the worker stays. In our model, as in Harris and Holmstrom (1982), there are no efficiency gains from separations. Adopting the convention that workers stay when (3) binds is without loss of generality and lets us avoid additional notation that would be needed to describe job transitions.[18]

[16] We can equivalently write $U(c_t, a_t)$ as $u(c_t + 1_{a_t=a_l}\log(\phi^{-1}))$ and interpret $\log(\phi^{-1}) > 0$ as the consumption equivalent of the utility the agent gets from leisure associated with exerting low effort.

[17] Note that because $\boldsymbol{y}$ is a Markov process, V does not directly depend on t. Also, since y_t is persistent and V is strictly increasing, the high-drift/high-effort action a_h improves both the output the worker produces for the firm and the worker's own market value at all future dates.

[18] If we follow the alternative convention and suppose that the worker quits when (3) binds, the optimal

Because action a_t is not observable, contracts will also have to satisfy incentive compatibility (IC) constraints. A contract is incentive compatible if no deviation from the recommended action process $\boldsymbol{a}$ can make the worker better off. We express IC constraints using the following representation result of Sannikov (2008).

Let $(\boldsymbol{a}, \boldsymbol{c})$ be a contract and $\boldsymbol{W}$ the associated continuation utility process as defined in (2). There exists a (progressively measurable) process $\boldsymbol{\Delta} = \{\Delta_t; t \geq 0\}$ such that the continuation utility process $\boldsymbol{W}$ can be represented as

$$dW_t = r(W_t - U(c_t, a_t))dt + \Delta_t dz_t^a, \tag{4}$$

where

$$z_t^a = \sigma^{-1}\left(y_t - y_0 - \int_0^t a_s ds\right). \tag{5}$$

Sannikov (2008) shows that contract $(\boldsymbol{a}, \boldsymbol{c})$ is IC if and only if for all t and $\tilde{a} \in \{a_h, a_l\}$,

$$r\left(U(c_t, \tilde{a}) - U(c_t, a_t)\right) + \sigma^{-1}(\tilde{a} - a_t)\Delta_t \leq 0. \tag{6}$$

Equation (4) represents the continuation utility process as a diffusion with drift $r(W_t - U(c_t, a_t))$ and sensitivity Δ_t.[19] The innovation $dz_t^a = \sigma^{-1}\left(dy_t - a_t dt\right)$ represents the worker's current on-the-job performance. Performance at t is measured by the change in the worker's output, dy_t, relative to what this change is expected to be at t under the recommended action, $a_t dt$, and normalized by σ. Note that as long as the worker follows the recommended action a_t, her (observable) performance dz_t^a will be the same as the (unobservable) innovation term dz_t in her productivity process given in (1). The drift term $r(W_t - U(c_t, a_t))$ gives the expected change in the continuation utility W_t. If $U(c_t, a_t) = W_t$ at all t, then W_t is a martingale. The term Δ_t represents the sensitivity of the worker's continuation utility to current performance. Clearly, a larger Δ_t will imply a stronger response of W_t to any given observed performance dz_t^a.

The IC constraint (6) requires that the total gain the worker can obtain by deviating from the recommended action a_t to the alternative action $\tilde{a}$ be nonpositive. The first component of this gain shows the direct impact of the deviation on the worker's current utility. The second component shows the indirect impact of the deviation on the continuation utility expressed as the product of the action's impact on the worker's performance and the sensitivity of the

contract is the same except it ends when (3) binds for the first time and is replaced with a new contract identical to the continuation of the original contract. This interpretation of long-term contracts is equivalent to the no-separation convention we adopt in that it leads to identical production, consumption, and welfare. Optimal contracting models in which terminations occur in equilibrium include Spear and Wang (2005), DeMarzo and Fishman (2007), and DeMarzo and Sannikov (2006).

[19]For any process $\{x_t; t \geq 0\}$ satisfying a diffusion equation $dx_t = \alpha_t dt + \sigma_t dz_t^a$, we will refer to α_t as the drift of x_t and to σ_t as the sensitivity of x_t.

continuation value to performance.

If the recommended action at time t is to exert effort, i.e., if $a_t = a_h$, then the IC condition (6) reduces to $ru(c_t)(\phi - 1) \leq \sigma^{-1}(a_h - a_l)\Delta_t$, or

$$\frac{\Delta_t}{-u(c_t)} \geq \beta, \tag{7}$$

where $\beta = r\sigma\frac{1-\phi}{a_h - a_l} > 0$. Analogously, the low-effort action a_l is incentive compatible at t if and only if

$$\frac{\Delta_t}{-u(c_t)} \leq \beta.$$

Written in this form, the IC constraints make it clear that the ratio $\Delta_t/(-u(c_t))$ measures the strength of effort incentives that contract $(\boldsymbol{a}, \boldsymbol{c})$ provides to the worker at time t. The high-effort action a_h is incentive compatible at t if and only if this ratio is greater than the constant β. Low effort is incentive compatible if and only if this ratio is smaller than β. As in Sannikov (2008), higher sensitivity of the worker's continuation value to her current on-the-job performance, Δ_t, makes effort incentives stronger. Due to non-separability of workers' preferences between consumption and leisure, the level of consumption c_t also affects the strength of effort incentives in our model.[20] In particular, if the contract recommends high effort, the gain in the flow utility the worker can obtain by shirking is in our model smaller at higher consumption levels.[21] For a given level of sensitivity Δ_t, thus, higher current consumption c_t makes effort incentives stronger.

We are now ready to define the contract design problem faced by a firm that has just been matched with a worker. We will define this problem generally as a cost minimization problem in which the worker's current productivity level is $y \in \mathbb{R}$ and the firm needs to deliver to the worker some present discounted utility value $W \in [V(y), 0)$. Although in equilibrium the matching between workers and firms happens only at $t = 0$, the contract design problem is defined generally allowing the starting date to be any $t \geq 0$. Let $\Sigma(y)$ denote the set of all contracts $(\boldsymbol{a}, \boldsymbol{c}) = (\{a_s; s \geq t\}, \{c_s; s \geq t\})$ that satisfy quitting constraints (3) and IC constraints (6) for all $s \geq t$. The firm's minimum cost function $C(W, y)$ is defined as

$$C(W, y) = \min_{(\boldsymbol{a},\boldsymbol{c})\in\Sigma(y)} \quad \mathbb{E}^{\boldsymbol{a}}\left[\int_t^\infty re^{-r(s-t)}(c_s - y_s)ds\,|\mathcal{F}_t\right] \tag{8}$$

$$\text{subject to} \quad \mathbb{E}^{\boldsymbol{a}}\left[\int_t^\infty re^{-r(s-t)}U(c_s, a_s)ds\,|\mathcal{F}_t\right] = W. \tag{9}$$

[20]Compare our IC constraint (6) with the IC constraint (21) on page 976 of Sannikov (2008). Consumption c_t does not show up in the IC constraint of that model because preferences considered there are additively separable between consumption and effort.

[21]This property is particularly easy to see if we interpret $\log(\phi^{-1}) > 0$ as the consumption equivalent of the utility the agent gets from shirking. Since shirking at t is equivalent to consuming $c_t + \log(\phi^{-1})$ instead of c_t, decreasing marginal utility of consumption implies that the gain from shirking is lower when c_t is higher.

The constraint (9) is known as the promise-keeping constraint: the contract must deliver to the worker the initial value W. In the special case of $W = V(y)$, the value $-C(V(y), y)$ represents the profit the firm attains in a match with a worker of type y when the worker's outside value function is V. Note that conditional on (W, y), the cost function C does not depend on calendar time t. In the language of dynamic programming, (W, y) is a sufficient state variable fully characterizing the worker, and the firm's cost depends on this state variable only.

Next, we define competitive equilibrium in the labor market with long-term contracts.

Definition 1 *Competitive equilibrium consists of the workers' market value function* $V : \mathbb{R} \to \mathbb{R}_-$ *and a collection of contracts* $(\boldsymbol{a}^y, \boldsymbol{c}^y)_{y \in \mathbb{R}}$ *such that, for all* $y \in \mathbb{R}$,

(i) $(\boldsymbol{a}^y, \boldsymbol{c}^y)$ *attains the minimum cost* $C(V(y), y)$ *in the firm's problem* (8)–(9),

(ii) $C(V(y), y) = 0$ *and* $C(W, y) > 0$ *for any* $W > V(y)$.

The first equilibrium condition requires that when firms assume (correctly) that the workers' outside value is their equilibrium market value, then the equilibrium contracts are cost-minimizing (i.e., efficient) and in fact deliver to workers their market value. The second condition comes from perfect competition under free entry: profits attained by firms must be zero in equilibrium and no firm can deliver to a worker a larger value than her market value without incurring a loss.

2.3 Level-independence of incentives

The following proposition shows a simple relationship between the contracts that workers with different productivity levels receive at $t = 0$ in equilibrium. This relationship implies a particularly simple functional form for the equilibrium worker value function V and gives us a partial characterization of the firm's cost function C.

Proposition 1 *If* $(\boldsymbol{a}^0, \boldsymbol{c}^0)$ *is an equilibrium contract for a worker with* $y_0 = 0$, *then an equilibrium contract for a worker with any* $y_0 \in \mathbb{R}$, $(\boldsymbol{a}^{y_0}, \boldsymbol{c}^{y_0})$, *is given by*

$$\boldsymbol{a}^{y_0} = \boldsymbol{a}^0, \tag{10}$$

$$\boldsymbol{c}^{y_0} = \boldsymbol{c}^0 + y_0. \tag{11}$$

The equilibrium value function V *satisfies*

$$V(y) = e^{-y} V(0) \quad \forall y \in \mathbb{R}. \tag{12}$$

The minimum cost function C *satisfies*

$$C(W, y) = C(W e^y, 0) \quad \forall y \in \mathbb{R}, W < 0. \tag{13}$$

The independence of the optimal action recommendation from y_0, shown in (10), and the additivity of the optimal compensation plan with respect to y_0, shown in (11), follow from the independence of future productivity changes dy_t from the initial condition y_0 and from the absence of wealth effects in CARA preferences. With no wealth effects, incentives needed to induce high or low effort are the same for workers of all productivity levels. The contribution of changes in a worker's productivity to a firm's revenue is also the same for all workers. Thus, the same effort process is optimally recommended to workers of all productivity levels, and output produced by a worker with initial productivity $y_0 = y > 0$ is path-by-path larger by exactly y than output produced by a worker with initial productivity $y_0 = 0$. Competition among firms implies then that in equilibrium the worker with $y_0 = y$ will obtain the same compensation process as the worker with $y_0 = 0$ plus the constant amount y at all t.

This structure of the compensation plan allows us to pin down the functional form of the workers' market value function $V(y_0)$, as given in (12). Intuitively, if a worker with $y_0 = 0$ obtains $V(0)$ in market equilibrium, then a worker with $y_0 = y$ will obtain $e^{-y}V(0)$ because her consumption is larger by y at all t and the utility function is exponential, so $u(c_t+y) = e^{-y}u(c_t)$ at all t.

In addition, this structure of optimal contracts implies a particular form of homogeneity for a firm's minimum cost function $C(W, y)$, as shown in (13). Suppose some contract efficiently delivers some value $W < 0$ to a worker whose initial productivity $y_0 = y > 0$ (i.e., this contract attains $C(W, y)$). Then a modified contract with compensation uniformly decreased by y will efficiently deliver value $e^y W < W$ to a worker whose initial productivity $y_0 = 0$ (i.e., the modified contract will attain $C(e^y W, 0)$). But these two contracts generate the same cost/profit for the firm, as in the second case the worker produces less output (uniformly less by y) and receives less compensation (also less by y).[22]

The scalability of the contracting problem and the implied homogeneity of the minimum cost function greatly simplify our analysis in this paper. In order to solve for the equilibrium, it is sufficient to find one value, $V(0)$, and one contract that supports it, $(\boldsymbol{a}^0, \boldsymbol{c}^0)$.

2.4 Optimality of high effort

As we focus on the dynamics of compensation in this paper, we impose a sufficient condition for the high-effort action a_h to be optimal and therefore always used by firms in equilibrium. In the absence of information and commitment frictions, i.e., in the first best, the firm provides full insurance to the worker, i.e., it keeps the worker's utility constant. To keep $U(c_t, a_t)$ constant, the firm must pay higher compensation c_t if it requires the high-effort action a_h, because effort is costly to the worker. In particular, under action a_h compensation must be higher by $\log(\phi^{-1})$ than under action a_l. High effort at t, however, increases the worker's output permanently by

[22] Similarly, a worker with initial $y_0 = -y < 0$ will produce and receive y units less than a worker with $y_0 = 0$.

$a_h - a_l$. In the absence of frictions, therefore, high effort is optimal if and only if

$$a_h - a_l \geq r \log(\phi^{-1}). \tag{14}$$

With limited commitment and moral hazard, there is an additional cost of implementing the high-effort action a_h: under high effort the firm cannot insure the worker as well as under low effort. We will verify in Section 5 that the following modification of (14) is sufficient for the action a_h to be optimal at all times under moral hazard and one-sided commitment.

Assumption 1 *Let* $\kappa = \sigma^{-2}\left(\sqrt{a_h^2 + 2r\sigma^2} - a_h\right)$. *We assume that*

$$\frac{\kappa}{1+\kappa}(a_h - a_l) \geq r \log\left(\phi^{-1}\right) + \frac{1}{2}\beta\sigma. \tag{15}$$

In (15), the multiplicative factor $\kappa/(1+\kappa)$ represents the additional cost of implementing effort under limited commitment. High effort makes the worker's productivity grow faster, which increases the worker's upside risk. Because this risk is not fully insurable under limited commitment, implementing high effort becomes more costly in the presence of this friction. Similarly, $\beta\sigma/2$ represents the additional cost of implementing effort under moral hazard. This cost, as well, is positive because moral hazard restricts the firm's ability to insure the worker.[23]

It is not hard to check that (15) holds for low enough a_l, so the set of parameter values satisfying Assumption 1 is nonempty. We will maintain this assumption throughout the paper.

2.5 Recursive formulation

In order to find the cost function $C(W_t, y_t)$, we will use the methods of Sannikov (2008) to study a recursive minimization problem with control variables a_t, $u_t \equiv u(c_t)$, and Δ_t. Scalability and homogeneity properties of Proposition 1 let us reduce the dimension of the state space in this recursive problem. Instead of studying this problem in the two-dimensional state vector (W_t, y_t), we can reduce the state space to a single dimension as follows. Using (13) and (12), we have

$$C(W_t, y_t) = C(W_t e^{y_t}, 0) = C\left(\frac{W_t}{e^{-y_t}V(0)}V(0), 0\right) = C\left(\frac{W_t}{V(y_t)}V(0), 0\right). \tag{16}$$

This shows that the minimum cost $C(W_t, y_t)$ is the same for all pairs (W_t, y_t) for which the ratio $W_t/V(y_t)$ is the same. We will find it convenient to transform this ratio further and define a single state variable as

$$S_t \equiv \log\left(\frac{V(y_t)}{W_t}\right). \tag{17}$$

[23]In Sections 3 and 4, we discuss in detail how moral hazard and limited commitment, taken separately, impair the firm's provision of insurance.

Using S_t, we can express the firm's cost function as

$$C(W_t, y_t) = C\left(\frac{W_t}{V(y_t)}V(0), 0\right) = C\left(e^{-S_t}V(0), 0\right) = C\left(V(S_t), 0\right),$$

where the first equality uses (16), the second uses (17), and the third uses (12). We will denote $C(V(\cdot), 0)$ by $J(\cdot)$ and solve for this function in the state variable S_t.

To study the firm's cost minimization problem in S_t, we must express not only the objective function but also the constraints of this problem in terms of S_t. The IC constraint (7) is not affected by the change of the state variable because it depends on the control variables only. Using (17), we can express the worker's quitting constraint (3) as

$$S_t \geq 0. \tag{18}$$

Thus, S_t measures how slack the quitting constraint is at time t. When $S_t = 0$, the quitting constraint binds, i.e, the worker is indifferent between continuing with her current contract and quitting.

Note that S_t measures slackness in the quitting constraint in units of permanent consumption. Using the inverse utility function $u^{-1}(x) = -\log(-x)$, $x < 0$, we can express S_t as $u^{-1}(W_t) - u^{-1}(V(y_t))$. Thus, S_t is the difference between the worker's continuation value inside the contract and her outside market value when both these values are converted to permanent compensation equivalents. Indeed, if $S_t = S$ for some $S > 0$, then the worker is indifferent between giving up S units of her compensation forever and separating from the firm.[24]

With the worker equilibrium value function (12) substituted into (17), we can write the state variable S_t as

$$S_t = -\log(-W_t) - y_t + \log(-V(0)). \tag{19}$$

Using Ito's lemma, the law of motion for y_t given in (1), and the law of motion for W_t given in (4), we obtain the law of motion for the state variable S_t under high effort as

$$dS_t = \left(r\left(-1 - \frac{u_t}{-W_t}\right) + \frac{1}{2}\left(\frac{\Delta_t}{-W_t}\right)^2 - a_h\right)dt + \left(\frac{\Delta_t}{-W_t} - \sigma\right)dz_t^a. \tag{20}$$

The expected change in the slackness S_t consists of three terms. The first term accounts for the impact of the current utility flow u_t on the permanent compensation equivalent of the worker's continuation value W_t. In particular, if $u_t > W_t$, then the continuation value owed to the worker decreases, and so does its permanent compensation equivalent $-\log(-W_t)$. This, ceteris paribus, reduces slackness in the quitting constraint. The second term in the drift of

[24] To see this, note that if $S_t = S$ and $\{c_{t+s}; s \geq 0\}$ is a compensation process that gives the worker the continuation value W_t, then the compensation process $\{c_{t+s} - S; s \geq 0\}$ gives the worker the continuation value exactly equal to the value of her outside option, $V(y_t)$.

S_t, $\frac{1}{2}(\Delta_t/-W_t)^2$, accounts for the expected increase in permanent compensation associated with risk exposure Δ_t. For a given continuation value W_t to be delivered to the worker, larger Δ_t will require higher expected compensation (in permanent units) because the worker is risk averse. The third term comes from the expected change in the worker's productivity. Faster productivity growth increases the worker's market value, which decreases slackness in the quitting constraint, ceteris paribus. The sensitivity term in (20) is simply the difference between the sensitivities of $u^{-1}(W_t)$ and $u^{-1}(V(y_t))$. We will find it useful to normalize the control variables u_t and Δ_t by the absolute value of the worker's continuation utility. Introducing $\hat{u}_t \equiv \frac{u_t}{-W_t}$ and $\hat{\Delta}_t \equiv \frac{\Delta_t}{-W_t}$, we express (20) as

$$dS_t = \left(r\left(-1-\hat{u}_t\right)+\frac{1}{2}\hat{\Delta}_t^2-a_h\right)dt+\left(\hat{\Delta}_t-\sigma\right)dz_t^a. \tag{21}$$

The Hamilton-Jacobi-Bellman (HJB) equation for the firm's cost function J is

$$\begin{aligned} rJ(S_t) \quad = \quad & rS_t - r\log(-V(0)) + \min_{\hat{u}_t,\hat{\Delta}_t}\Bigg\{ r(-\log(-\hat{u}_t)) + \\ & J'(S_t)\left(r\left(-1-\hat{u}_t\right)+\frac{1}{2}\hat{\Delta}_t^2-a_h\right)+\frac{1}{2}J''(S_t)\left(\hat{\Delta}_t-\sigma\right)^2\Bigg\}, \end{aligned} \tag{22}$$

where control variables must satisfy $\hat{\Delta}_t \geq -\hat{u}_t\beta$ to ensure incentive compatibility of the recommended high-effort action a_h. Note that both the HJB equation and its solution J depend on the value $V(0)$, which in equilibrium is determined by the firm break-even condition $J(0)=0$. The meaning of the terms in the HJB equation is standard. It may be helpful to write the HJB equation informally as

$$rJ(S_t)=\min\left\{r(c_t-y_t)+J'(S_t)\,(\text{drift of } S_t)+\frac{1}{2}J''(S_t)\,(\text{sensitivity of } S_t)^2\right\}. \tag{23}$$

Intuitively, the first derivative J' represents the firm's aversion to the drift of S_t because, as we see in (23), the total cost $rJ(S_t)$ increases by $J'(S_t)$ when the drift of S_t increases by one unit. Similarly, the second derivative J'' shows how strongly the cost function will respond to an increase in the volatility of S_t, so in this sense it represents the firm's volatility aversion. Also, using definitions of S_t and $\hat{u}_t$, it is easy to verify that the first three terms on the right-hand side of (22) represent the firm's flow cost $r(c_t-y_t)$.

In Section 5, we will characterize optimal long-term contracts by finding a unique solution to the HJB equation subject to appropriate boundary and asymptotic conditions. In the next two sections, we provide two important benchmarks by finding optimal contracts in simplified versions of our general environment in which one of the two frictions is absent.

3 Pay-for-performance incentives in equilibrium with private information and full commitment

In this section, we will assume full commitment: not only firms but also workers have the power to commit to never breaking the contract. As in our general model presented in the previous section, firms match with workers and offer them long-term contracts at $t = 0$. At this time, the worker can reject the offer and move to another match instantaneously. Upon accepting a contract at $t = 0$, however, the worker commits to not quitting at any $t > 0$. This commitment maximizes the match's surplus as it allows firms to provide better insurance against fluctuations in workers' productivity relative to the case in which the workers would not commit. In particular, it lets firms insure the upside risk to workers' productivity. We solve this version of our model in closed form. In equilibrium, firms provide incentives to workers by making compensation sensitive to current on-the-job performance.

Let $\Sigma_{FC}(y_0)$ denote the set of all contracts $(\boldsymbol{a}, \boldsymbol{c})$ that at all t satisfy the IC constraint (6). The contracting problem we study in this section is identical to the cost-minimization problem in (8) but with the quitting constraint (3) removed, i.e., with the set of feasible contracts expanded from $\Sigma(y_0)$ to $\Sigma_{FC}(y_0)$. We will use $C_{FC}(W, y_0)$ to denote the minimum cost function in this problem. The reduced-form cost function $J_{FC}(S)$ is defined analogously. Note that $J_{FC}(S)$ is defined for any S, even negative. Market equilibrium is defined as in the general case but using the cost function $C_{FC}(W, y_0)$ instead of $C(W, y_0)$.

The following proposition is a closed-form version of standard characterization results for optimal contracts with private information and full commitment, e.g., Spear and Srivastava (1987), Thomas and Worrall (1990).

Proposition 2 *In the model with full commitment, workers' equilibrium compensation is given by*

$$c_t = y_0 + \frac{\mu + a_h}{r} - \mu t + \rho \beta z_t^a, \tag{24}$$

where $0 < \rho = (\sqrt{1 + 4r^{-1}\beta^2} - 1)/(2r^{-1}\beta^2) < 1$ *and* $\mu = r(1 - \rho) - \frac{1}{2}\rho^2\beta^2 > 0$. *The sensitivity of the equilibrium continuation value* W_t *with respect to observed performance* dz_t^a *is*

$$\Delta_t = -u(c_t)\beta \quad \text{at all } t. \tag{25}$$

Proposition 2 shows two main features of optimal compensation schemes in the model with private information and full commitment: contemporaneous sensitivity of compensation to performance, represented in (24) by $\rho\beta > 0$, and a negative time trend in compensation, represented by $-\mu < 0$. The positive contemporaneous sensitivity of compensation with respect to the worker's observed performance is the standard, contract-based, "pay-for-performance" incentive for workers to exert effort. The negative trend in compensation does not provide effort incentives by itself, but it improves the effectiveness of the pay-for-performance incentive.

The sensitivity of the worker's continuation value to her performance, given in (25), shows that in equilibrium with private information and full commitment the IC constraint (7) binds at all t. This means that incentives provided to the worker, measured by the ratio $\Delta_t/(-u(c_t))$, are in equilibrium strong enough to make the recommended high-effort action a_h incentive compatible but not any stronger. Incentives are costly because they reduce insurance. The equilibrium contract is efficient in holding incentives down to a necessary minimum at all times. Because this minimum does not change over time, the strength of incentives provided to the worker is always the same, i.e., β, in this model.

This section shows that private information requires positive sensitivity Δ_t. The next section shows that positive sensitivity Δ_t can arise completely independently of private information: if workers lack commitment, their productivity shocks cannot be fully insured and, therefore, their continuation values must remain sensitive to realizations of these shocks. Thus, in an environment in which private information and limited commitment coexist, limited commitment potentially could deliver the positive sensitivity Δ_t that private information requires. Our main results in this paper, which we give in Section 5, consider precisely this possibility.

4 Market-based incentives in equilibrium with limited commitment and full information

In this section, we discuss the full-information version of our model. As in the general model outlined in Section 2, firms match with workers and offer them long-term contracts at $t = 0$. A worker who has accepted a contract retains the option to quit and go back to the labor market, where she can find a new match instantaneously. Unlike in the general model, however, we will assume in this section that workers' actions on the job are observable, and that workers can contractually commit to a prescribed course of action.[25] The model we study in this section is essentially a continuous-time version of the Harris and Holmstrom (1982) model of rigid wages. This section also generalizes the optimal insurance model studied in Grochulski and Zhang (2011), where the outside option is assumed to be autarky.

Let $\Sigma_{FI}(y_0)$ denote the set of all contracts $(\boldsymbol{a}, \boldsymbol{c})$ that at all t satisfy the quitting constraint (3). The contracting problem we study in this section is identical to the cost-minimization problem in (8) but with the IC constraint (7) removed, i.e., with the set of feasible contracts expanded from $\Sigma(y_0)$ to $\Sigma_{FI}(y_0)$. We will use $C_{FI}(W, y_0)$ to denote the minimum cost function in this problem. The reduced-form cost function $J_{FI}(S)$ is defined analogously. Market equilibrium is defined as in the general case but using the cost function $C_{FI}(W, y_0)$ instead of $C(W, y_0)$.

[25] In short, workers cannot be punished for quitting but can be punished for shirking on the job.

Proposition 3 *In the model with full information, workers' equilibrium compensation is given by*

$$c_t = m_t - \psi, \tag{26}$$

where $m_t = \max_{0 \le s \le t} y_s$ *and* $\psi = \frac{\kappa\sigma^2}{2r} > 0$*. The sensitivity of the continuation value* W_t *with respect to observed performance* dz_t^a *is*

$$\Delta_t = -u(c_t)\frac{\kappa}{\kappa+1}e^{-\kappa(m_t - y_t)}\sigma > 0. \tag{27}$$

As in Grochulski and Zhang (2011), the distance between current productivity y_t and the maximum level that productivity has attained to date, m_t, is a measure of slackness in the quitting constraint. The quitting constraint binds whenever productivity attains a new to-date maximum, i.e., when $y_t = m_t$, and is slack whenever productivity is below its to-date maximum, i.e., when $y_t < m_t$.[26]

As we see in (26), equilibrium compensation follows the downward-rigid wage pattern of Harris and Holmstrom (1982): compensation is constant unless the quitting constraint binds; when it binds, compensation increases monotonically. Thus, compensation never decreases, and it increases faster the faster new all-time high levels of a worker's productivity are attained. The mechanism behind this wage structure is the same as in Harris and Holmstrom (1982): if an outside firm could hire the worker away from the current firm and make a positive profit, the worker's wage is bid up; otherwise, the wage is constant as the current employer absorbs all downside risk in the worker's productivity process.

Since sample paths of the productivity process are continuous, a worker has a better chance of attaining a new to-date maximum of her productivity—and thus obtaining a permanent increase in her compensation—the closer her current productivity level y_t is to the current to-date maximum m_t. The worker's continuation value in the contract, W_t, increases whenever the chance for the next permanent increase in compensation improves. This means that W_t increases whenever current productivity y_t increases, even during time intervals in which y_t remains strictly below m_t, i.e., when current compensation c_t does not at all respond to changes in y_t. This everywhere-positive sensitivity of the continuation value to current performance is shown in (27). Moreover, (27) shows that the continuation value's performance sensitivity Δ_t increases as the distance between y_t and m_t decreases. Thus, sensitivity Δ_t is larger the closer the quitting constraint is to binding.

Sensitivity Δ_t is positive here and in the full-commitment model discussed in the previous section, but for completely different reasons. There, firms pay for performance in order to elicit effort. Here, firms can directly control workers' effort, but face the possibility of workers

[26]In fact, the distance $m_t - y_t$ is isomorphic to our state variable S_t with $S_t = m_t - y_t - \log\left(\kappa + 1 - e^{-\kappa(m_t - y_t)}\right) + \log(\kappa)$. Clearly, S_t is strictly increasing in $m_t - y_t$, and $S_t = 0$ if and only if $m_t - y_t = 0$.

quitting. When the quitting constraint becomes binding, the firm must give the worker a raise in order to retain her. This raise is the source of positive sensitivity of the continuation value to current performance at all times, even when the quitting constraint is slack. Because the source of sensitivity Δ_t in this section is the worker's market option, we will call this Δ_t market-induced sensitivity.

As we see in the IC constraint (7), incentives are measured in our model by the ratio of Δ_t to $-u(c_t)$. Sensitivity Δ_t is therefore closely related to the notion of incentives. Despite there being no need for incentives here, as we assume in this section that effort is observable and contractually controllable, we should note that the contract in Proposition 3 still gives the worker an effort incentive because the ratio $\Delta_t/(-u(c_t))$ is nonzero. Indeed, if the firm were to not observe the worker's effort for a short instant starting at time t, the worker would still choose to supply effort at t as long as the ratio $\Delta_t/(-u(c_t))$ is larger than β. Thus, even without moral hazard, an effort incentive exists here just because sensitivity Δ_t is positive. Since this sensitivity is market-induced, we will call this incentive the market-based incentive.

Corollary 1 *The ratio $\frac{\Delta_t}{-u(c_t)}$ is strictly decreasing in $m_t - y_t$. In particular, $\frac{\Delta_t}{-u(c_t)} \geq \beta$ if and only if $m_t - y_t \leq \delta$, where $\delta = \kappa^{-1} \log\left(\frac{\kappa}{\kappa+1}\frac{\sigma}{\beta}\right) > 0$.*

This corollary shows that the equilibrium contract obtained in the full-information model formally satisfies the IC constraint (7) whenever slackness in the quitting constraint (3), as measured by $m_t - y_t$, is small. That means that the market-based incentive is strong in this region.[27] The corollary also shows that the full-information contract is not overall incentive compatible because it fails to satisfy the IC constraint (7) when the quitting constraint is sufficiently slack. Monotonicity of $\Delta_t/(-u(c_t))$ in $m_t - y_t$ means that the market-based incentive is stronger when the slack in the quitting constraint is smaller.

In this section, there is no need for incentives. Yet, they exist in equilibrium as a by-product of limited commitment. In the next section, we consider the general version of our model with both moral hazard and limited commitment, where incentives are needed. There, as here, the market option improves with the worker's performance, which will generate a market-based incentive. Similar to Corollary 1, the market-based incentive will be strong (sufficient to induce high effort) when slackness in the quitting constraint is smaller than a threshold. In that region of the state space, therefore, the equilibrium contract will rely completely on market-based incentives and will not use pay-for-performance incentives at all.

4.1 Further properties of equilibrium with full information

Proposition 3 describes the equilibrium contract in the full-information model using two state variables: m_t and y_t. In the Appendix, we describe the equilibrium of this model in terms

[27] In particular, the full-information equilibrium contract does satisfy the IC constraint at the onset of every employment relationship because $m_0 - y_0 = 0 < \delta$.

of the single state variable S_t, and characterize the cost function $J_{FI}(S_t)$. In particular, we show the following dynamic properties of the state variable under the equilibrium contract. The drift and the sensitivity of S_t are strictly decreasing in S_t. The possibility of violating the quitting constraint makes the firm infinitely averse to volatility in S_t at the boundary $S = 0$. Hence, $J''_{FI}(0) = \infty$ and the sensitivity of S_t at $S = 0$ is zero in equilibrium. The drift of S_t at $S = 0$ is strictly positive, i.e., $S = 0$ is a reflective barrier for the state variable S_t. In the next section, we show that all these properties continue to hold when both private information and limited commitment are present in the model.

5 Market-based and pay-for-performance incentives in equilibrium with both frictions

In this section, we characterize the optimal contract in our general model, where firms face both the incentive problem and the quitting constraint.

5.1 Solving the optimal contracting problem

Standard methods for solving second-order differential equations like our HJB equation (22) require two boundary conditions. Our problem is nonstandard. It has a semi-unbounded domain (the positive half-line) with only one boundary condition: the second derivative of J at the boundary $S_t = 0$ must be infinite because otherwise the quitting constraint would be violated immediately after S_t becomes zero. Despite the lack of a second condition on J at the boundary, our analysis of the full-information model suggests an asymptotic condition that can be used to pin down the solution: the cost that the quitting constraint imposes on the firm must become negligible when S_t goes to infinity because the (time-discounted) chance of the constraint binding in the future becomes negligible when S_t is large. When S_t goes to infinity, therefore, the cost function in the model with two frictions, J, must converge to the cost function from the model with private information and full commitment, J_{FC}. In particular, first derivatives of these functions, $J'(S_t)$ and $J'_{FC}(S_t)$, must become close at large values of S_t. We will use this asymptotic convergence condition to pin down the solution.[28] [29]

Our analysis of the HJB equation gives the following theorem.

[28] In order to use the cost function J_{FC} from the one-friction model with full commitment as a benchmark (lower bound) for J in the two-friction model, one must shift J_{FC} downward by a constant to account for the fact that a lower level of utility is provided to the worker in equilibrium in the model with two frictions (the value $V(0)$ is lower in this model). It is thus more convenient to express the asymptotic convergence condition in terms of first derivatives rather than levels because a uniform vertical shift of J_{FC} does not affect its derivative.

[29] In the Appendix, we discuss the cost that the quitting constraint imposes on the firm in the full-information model relative to the environment with no frictions (the first best). This cost does go to zero when slackness in the quitting constraint goes to infinity: the cost function J_{FI} and its derivatives converge to the first-best cost function and its derivatives, respectively.

Theorem 1 *There exists a unique solution to the HJB equation* (22) *satisfying the boundary condition* $J''(0) = \infty$ *and the convergence condition* $\lim_{S_t \to \infty}(J'(S_t) - J'_{FC}(S_t)) = 0$. *This solution represents the true minimum cost function for the firm.*

The method of proof given in the Appendix is similar to that in Sannikov (2008) with two technical difficulties stemming from the specific boundary and convergence conditions we have. First, our HJB equation does not satisfy the Lipschitz condition at $S_t = 0$ because $J''(0) = \infty$. We overcome this difficulty by using a change of variable technique. Second, the asymptotic condition requiring convergence of $J'(S_t)$ to $J'_{FC}(S_t)$ does not provide an actual restriction on the boundary of the state space. We overcome this difficulty as follows. We determine a range of possible values for the first derivative of J at $S_t = 0$ and consider a family of candidate solutions to the HJB equation, one for each possible value of $J'(0)$ in this range. We show that the asymptotic condition requiring that $J'(S_t)$ converge to $J'_{FC}(S_t)$ as $S_t \to \infty$ is violated by all but one candidate solution. We then confirm that the one candidate solution that satisfies this asymptotic condition indeed represents the true minimum cost function J.

Lastly, we verify that the recommendation of high effort is optimal at all t. Lemma A.11 in the Appendix shows that this conclusion follows from our Assumption 1.

5.2 The structure of the equilibrium contract

Proposition 4 *In the model with two frictions, there exists a unique* $S^* > 0$ *such that the IC constraint* (7) *binds whenever* $S_t \geq S^*$, *but is slack whenever* $S_t < S^*$. *In each time interval in which* S_t *remains strictly above* 0 *and below* S^*, *equilibrium compensation* c_t *is constant. When* S_t *hits zero, compensation increases monotonically with current productivity.*

This proposition shows that, despite moral hazard, optimal compensation in our model replicates the rigid-wage compensation structure of Harris and Holmstrom (1982) as long as slackness in the quitting constraint remains below a threshold. Intuitively, when quitting is near, moral hazard does not matter. The worker's exposure to her own performance risk is large enough to guarantee her full effort. In fact, a slack IC constraint means that the worker's incentives are too strong (i.e., more than necessary to induce effort) in the rigid-wage region $S_t < S^*$. The contract would be more efficient (i.e., of higher value to the worker) if the firm could provide more insurance, thus weakening her incentives. Doing so, however, is impossible, because the worker's right to quit makes her upside performance risk impossible to insure fully.

Standard models in the principal-agent literature with private information, e.g., the model in our Section 3, predict that pay should be volatile, i.e., at all times responding to the worker's performance. Proposition 4 shows that this is no longer true if external, market-based incentives are present. In fact, whenever the worker's market value is close to the value she obtains by continuing to work for the current employer, optimal compensation is constant, i.e., completely unresponsive to current performance of the worker, and the worker nevertheless chooses to supply effort.[30] Key to this result are two facts. First, as we have seen in Section 4, when the quitting constraint binds, the firm must increase the worker's compensation in order to retain her. Second, when the quitting constraint does not quite bind but is close to binding, the worker's effort has a strong impact on the chance that the quitting constraint becomes binding. These two facts imply that when the worker is close to quitting, she will expend effort in order to actually hit the quitting constraint and obtain a raise. Knowing this, the firm does not need to provide an additional incentive via performance-dependent compensation; constant compensation is efficient.

When the quitting constraint is relatively slack, $S_t > S^*$, the IC constraint binds. This is because the impact of the worker's effort on her chance of hitting the quitting constraint is smaller when the quitting constraint is more distant. The market option still gives the worker an incentive to supply effort, but this incentive is weak (i.e., not sufficient to induce effort). The firm must in this case supplement the market-based incentive with a contract-based pay-for-performance incentive. We study the optimal mix of these incentives in the next subsection. In the limiting case with $S_t \to \infty$, the chance of S_t ever returning to zero becomes negligible and the strength of market-based incentives goes to zero.

In sum, when S_t remains below S^* the optimal contract looks exactly like the optimal contract from the model with limited commitment and full information in Section 4. When S_t goes to infinity, in contrast, the optimal contract looks like the optimal contract from the model with private information and full commitment in Section 3.

5.3 Strength of market-based incentives

In our model, the strength of effort incentives provided to a worker at time t is measured by the ratio $\Delta_t/(-u(c_t))$. Workers will supply effort if and only if this ratio is larger than β. Proposition 4 shows that in equilibrium, the strength of incentives is only just sufficient to induce effort when the quitting constraint is relatively slack ($S_t \geq S^*$), but is more than sufficient when the quitting constraint is relatively tight ($S_t < S^*$).

We will now decompose incentives into two parts: external, market-based and direct, contract-based. Market-based incentives will be those induced by the worker's outside option (as in Section 4). Contract-based incentives will be those not induced by the market option (as in Section 3). To measure the strength of market-based incentives at t, we need to compute the ratio $\Delta_t/(-u(c_t))$ that the firm would optimally choose at t if limited commitment were the only friction, i.e., as if the worker's effort were observable (and hence controllable) by the firm

[30] He (2012) obtains downward-rigid compensation as a part of an optimal contract in a moral hazard model with Poisson uncertainty, zero probability of success under shirking, and private savings. Also, in Sannikov (2008), flat compensation arises in some specifications of a moral hazard model in which the agent must be inefficiently retired at a lower bound of the set of feasible continuation values. These mechanisms are different from ours as they rely exclusively on contract-based incentives, while the source of incentives giving rise to downward-rigid compensation is external in our model. In particular, the IC constraint binds at all times (prior to termination/retirement) in these two studies but it does not always bind in our model, and compensation is downward-rigid in our equilibrium only when the IC constraint is slack.

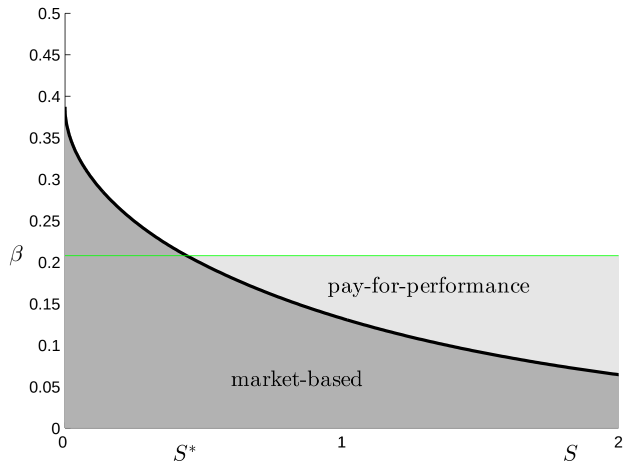


Figure 1: Composition of incentives.

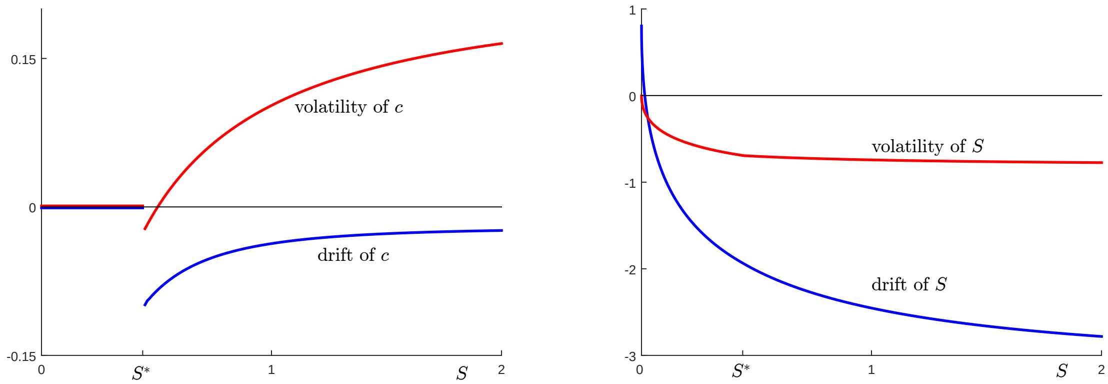
locally at t. We compute this ratio as follows. Given the optimal cost function J, we disregard the IC constraint at t in the HJB equation (22) and use first-order conditions to obtain current utility $u(c_t)$ and sensitivity Δ_t that the firm would choose in such a relaxed problem. Denoting the ratio $\Delta_t/(-u(c_t))$ from this locally relaxed problem by $\tilde{\Delta}_t/(-\tilde{u}(c_t))$, we have

$$\frac{\tilde{\Delta}_t}{-\tilde{u}(c_t)} = \frac{\sigma J'(S_t) J''(S_t)}{J'(S_t) + J''(S_t)}.$$

This ratio gives the portion of the actual $\Delta_t/(-u(c_t))$ that is induced by the presence of the worker's market option. Thus, it represents the strength of market-based incentives at t in our model. The remainder of the actual $\Delta_t/(-u(c_t))$ represents contract-based incentives that the firm must inject in order to ensure incentive compatibility of high effort at t.

Figure 1 plots the ratio $\tilde{\Delta}_t/(-\tilde{u}(c_t))$ against S_t in a numerical example.[31] The strength of market-based incentives decreases as the quitting constraint becomes more distant. Below S^*, market-based incentives are strong, meaning they are sufficient to induce effort, i.e., $\tilde{\Delta}_t/(-\tilde{u}(c_t)) \geq \beta$, and contract-based incentives are zero. An implication of strong market-based incentives when $S_t < S^*$, as we have seen in Proposition 4, is that compensation is flat and workers provide effort without being compensated for current performance. Above S^*, market-based incentives are weak, i.e., not strong enough to induce worker effort, and the optimal contract supplements them with pay-for-performance incentives. This means that compensation does depend on current performance above S^*. Pay-for-performance incentives become stronger and market-based incentives become weaker as the quitting constraint becomes more slack.

[31] Parameter values used in this example are $a_h = 3, a_l = 0, \phi = 0.7, r = 2.1, \sigma = 1$.



(a) Dynamics of compensation.

(b) Dynamics of S_t.

Figure 2: Example with $a_h > 0$. Threshold $S^* = 0.44$. Stationary point for S_t is 0.016.

5.4 Dynamics of the equilibrium contract

Unlike the two single-friction models studied in Sections 3 and 4, the model with both frictions does not admit a closed-form solution. In this section, therefore, we describe the dynamics of the equilibrium contract by characterizing the drift and the sensitivity of compensation c_t and the state variable S_t. We provide a mix of analytical and numerical results in this section. We start out by presenting in Figure 2 the drift and the sensitivity of c_t and S_t computed numerically under the parametrization of our model used earlier to produce Figure 1.

Dynamics of compensation. In panel (a), we can identify the region of strong market-based incentives by noting that for all S_t above zero and below S^* the drift and the sensitivity of compensation are both zero, which means that $dc_t = 0$, i.e., compensation remains constant in this region, as predicted earlier in Proposition 4. When S_t goes to infinity, the impact of the quitting constraint vanishes and optimal compensation converges to the optimal compensation from the full-commitment model, where, by Proposition 2, the drift of c_t is $-\mu < 0$ and the sensitivity of c_t is $\rho\beta > 0$. The main message from panel (a) of Figure 2, therefore, is that market-based incentives reduce the performance-sensitivity of compensation.

In addition to the properties of compensation at low and high values of the state variable, where market-based incentives are respectively strong and negligible, numerical analysis lets us characterize the dynamics of c_t in the intermediate region of the state space, where market-based incentives are not strong but are not negligible either. As we see in panel (a), at all S_t greater than S^* the sensitivity of compensation is increasing in S_t but remains smaller than its asymptotic value of $\rho\beta$. The intuition for this follows from the monotonicity of the strength of market-based incentives in S_t shown earlier in Figure 1. If at some $S_t > S^*$ the worker's observed performance is positive, $dz_t^a = dy_t - a_h dt > 0$, then both the worker's continuation value inside the contract and her outside market value increase. Because the contract provides some insurance to the worker, the outside market value increases by more than does the continuation value inside the contract. This means that the quitting constraint becomes less slack (S_t decreases) and, thus, the chance of entering the area of constant compensation (below S^*) and eventually hitting the quitting constraint (when the worker receives a raise) improves. This improvement provides some incentive for the worker to supply effort. Therefore, even in the region of weak market-based incentives, compensation can be less sensitive to contemporaneous performance than in the model with just moral hazard, where market-based incentives are absent. Because, as shown in Figure 1, the market-based incentive is stronger at smaller S_t, the sensitivity of c_t to performance decreases when S_t decreases at all $S_t > S^*$.

Panel (a) of Figure 2 shows that at $S_t = S^*$ (and, by continuity, also right above S^*), the sensitivity of compensation to observed performance is actually negative. This feature of the optimal contract is due to the non-separability in the worker's preferences between consumption and leisure.[32] The intuition for this is as follows. As we see in (7), higher current compensation c_t relaxes the IC constraint in our model. When the IC constraint binds, the firm saves on incentive costs by paying higher compensation now. If the IC constraint does not bind, this effect is absent. At the threshold point $S_t = S^*$, positive and negative worker productivity shocks dz_t have an asymmetric effect on the incentive benefit of high current compensation: positive shocks decrease S_t and take it into the region in which the IC constraint does not bind, where high current compensation is not needed, while negative shocks increase S_t and take it into the region where the IC constraint binds, where high current compensation does have a benefit. This produces negative sensitivity of compensation c_t to innovations in z_t at $S_t = S^*$: a positive shock $dz_t > 0$ will not affect c_t and a negative shock $dz_t < 0$ will increase c_t.

In addition, panel (a) of Figure 2 shows that the drift of compensation is lower than its asymptotic value of $-\mu$ at all S_t above S^*, and is monotonic in S_t in this region. Similar to the negative sensitivity of compensation, these properties of its drift are due to the fact that compensation increases when the state variable crosses the S^* threshold and enters the region of weak market-based incentives. A more strongly negative drift in c_t right above S^* helps average out the monotonic increase in c_t occurring at S^* as the state variable fluctuates around this threshold level. When S_t grows and moves away from S^*, this need for a more strongly negative drift vanishes and the drift in c_t approaches $-\mu$.

These dynamic properties of compensation are robust in the numerical experiments with the model we conducted. The discontinuity in the drift and the sensitivity of c_t at S^* can be shown analytically, but we do not have analytical results for the monotonicity of the drift and the sensitivity of c_t above S^*.

Dynamics of the state variable. Moving on to the dynamics of the state variable $S_t = u^{-1}(W_t) - u^{-1}(V(y_t))$, we note in panel (b) of Figure 2 that the sensitivity of S_t is everywhere negative and monotonically increasing toward zero as S_t decreases toward the boundary $S = 0$.

[32] In numerical examples with separable preferences that we computed, the sensitivity of consumption is everywhere weakly positive. It is zero at all S_t below S^* and positive at all S_t above S^*.

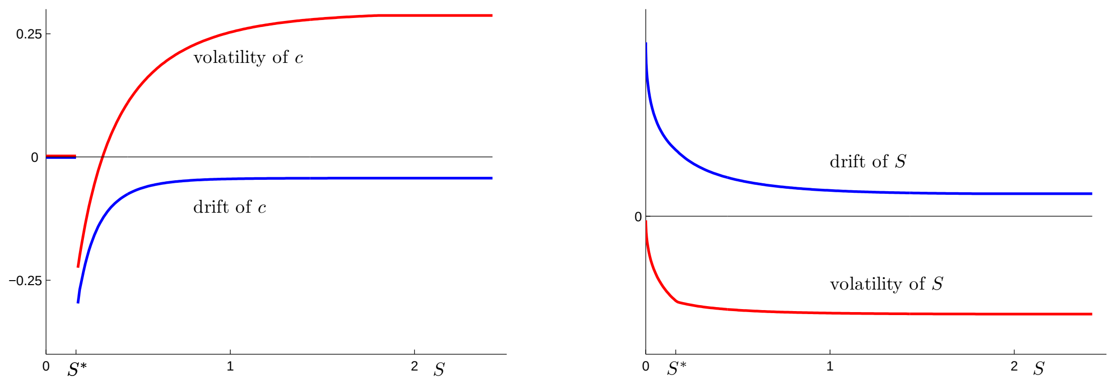
The intuition for this property follows from the fact that the optimal contract provides more insurance to a worker who is further away from quitting. At the boundary itself, the contract cannot provide any insurance, i.e., the performance-sensitivity of the continuation value W_t has to match the performance-sensitivity of the worker's outside option $V(y_t)$ in order to ensure that the quitting constraint is not violated immediately after the state variable hits its lower bound $S = 0$. The farther away S_t is from zero, the less likely it is that the quitting constraint becomes binding, the more fully the firm can insure the worker, and, in effect, the more negative the sensitivity of S_t becomes. Asymptotically, the sensitivity of S_t converges to its value from the model without quitting constraints.

The negative sensitivity of slackness S_t in the quitting constraint (3) means that this constraint can become binding only after the worker's good performance, which is exactly opposite to the standard moral hazard model without external incentives, e.g., Sannikov (2008). In both models, poor performance decreases the worker's continuation value W_t. In Sannikov (2008), the lower bound on W_t is fixed, so when W_t decreases, the distance between W_t and its lower bound decreases. In our model, the lower bound on W_t, $V(y_t)$, is not fixed: it is strictly increasing in y_t. In fact, $V(y_t)$ responds to the worker's performance more strongly than W_t. When performance is poor, thus, $V(y_t)$ decreases faster than W_t, so the distance between W_t and its lower bound increases. When performance is strong, $V(y_t)$ increases faster than W_t, i.e., the lower bound "catches up" to the continuation value W_t. The closer $V(y_t)$ approaches W_t, the slower this catching up becomes. When slackness S_t in the quitting constraint is zero, W_t and $V(y_t)$ respond to good performance exactly the same (S_t has zero sensitivity), so $V(y_t)$ "pushes up" W_t but never exceeds it.[33]

The drift of S_t, shown also in panel (b) of Figure 2, is positive at the boundary of the state space $S = 0$ and converges to its value from the model without quitting constraints when S_t goes to infinity. Clearly, drift of S_t must be nonnegative at zero or else the quitting constraint would be violated shortly after S_t hits zero. When S_t is large, the contract behaves as in the full commitment case, which determines the value of the drift of S_t in this region of the state space.

Moreover, note in Figure 2 that the drift of S_t at $S = 0$ is not only nonnegative, which is necessary to avoid violating the quitting constraint, but is actually strictly positive. Combined with the fact that S_t has zero sensitivity at $S = 0$, this implies that zero is a reflective barrier for the state variable in equilibrium. Although unable to provide insurance to the worker when S_t is at its lower bound, by paying compensation c_t lower than the worker's output y_t and increasing the worker's continuation value W_t, the firm can generate a positive drift in S_t when $S_t = 0$, which allows it to provide insurance to the worker in the future. This property of our model with market-based incentives is different from the absorbing lower bound that appears

[33]The moving lower bound $V(y_t)$ also highlights the difference between our model and models in which the contract is terminated at a fixed lower bound, e.g., Spear and Wang (2005), DeMarzo and Fishman (2007), and DeMarzo and Sannikov (2006).



(a) Dynamics of compensation. (b) Dynamics of S_t.

Figure 3: Example with $a_h < 0$. Threshold $S^* = 0.18$. No stationary point for S_t exists, i.e., the drift in S_t is everywhere positive.

in many dynamic moral hazard models with a fixed lower bound on the continuation utility, e.g., in Sannikov (2008).

These properties of the drift and the sensitivity of the state variable hold not only in the numerical example presented in Figure 2 but are true in our model in general. Formally, we have the following result.

Proposition 5 *Let $\alpha(S_t)$ and $\zeta(S_t)$ denote, respectively, the drift and the sensitivity of the state variable. In the equilibrium contract, $\alpha(S_t)$ is strictly decreasing with $\alpha(0) > 0$ and $\lim_{S_t \to \infty} \alpha(S_t) = -\mu - a_h$, and $\zeta(S_t)$ is strictly decreasing with $\zeta(0) = 0$ and $\lim_{S_t \to \infty} \zeta(S_t) = \rho\beta - \sigma$.*

Note that Proposition 5 implies that the sensitivity of S_t is always negative, but the sign of the drift in S_t is not pinned down. In particular, the direction in which S_t tends to move when it is large depends on the sign of $-\mu - a_h$. This value represents the drift of the state variable in the full-commitment version of our model. In the example presented in Figure 2, $-\mu - a_h < 0$ and the state variable has a unique stationary point, where its drift is zero. Because this stationary point is much smaller than S^* in this example, S_t tends to start to decrease toward zero before it reaches S^*, and thus it will only infrequently leave the region of strong market-based incentives.

The numerical example shown in Figure 3 modifies the parametrization used in Figure 2. In particular, in the modified parametrization the drift of the worker's productivity, a_h, is negative, and the asymptotic value for the drift of the state variable, $-\mu - a_h$, is positive.[34] Since, by Proposition 5, the drift of S_t is strictly positive at zero and monotonic in S_t, $-\mu - a_h > 0$ means that S_t has in this example positive drift everywhere in the state space. In this modified

[34]Parameter values used in this example are $a_h = -0.2, a_l = -2.2$, $\phi = 0.37, r = 1, \sigma = 1$.

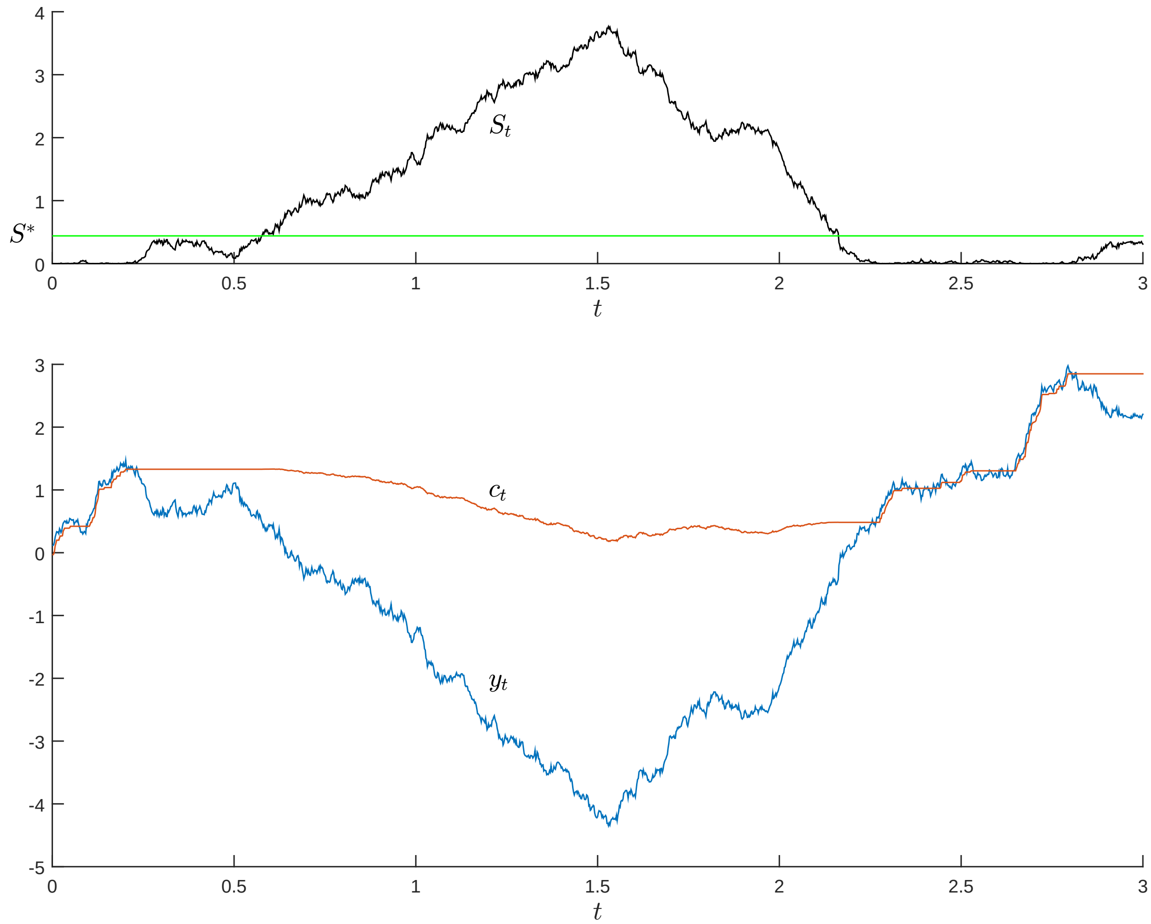


Figure 4: A sample career path along which the contract is in the rigid-wage phase before $t = 0.58$ and after $t = 2.16$. Parameters used in this figure are as in Figures 1 and 2.

parametrization, therefore, S_t tends to drift out of $(0, S^*)$. Over time, it thus becomes less and less likely that market-based incentives are strong: market-based incentives are transient in this parametrization.[35] These observations lead us to investigate where the state variable tends to spend most time in equilibrium. We provide these results in Section 5.6.

5.5 Sample career path

Having characterized the optimal contract in terms of the drift and sensitivity functions governing the dynamics of the state variable S_t and the compensation process c_t in the previous section, in this section we briefly discuss the worker's career path under the optimal contract. Figure 4 plots one sample path of the worker's output y_t, compensation c_t, and the state variable

[35]Panel (a) of Figure 3 shows that dynamic properties of compensation in the parametrization with low a_h are qualitatively the same as those presented in panel (a) of Figure 2 for the case of high a_h.

S_t generated under the parameters used earlier in Figures 1 and 2.

Before we discuss the worker's career path in this simulation, we should mention two features of the contract that are salient in Figure 4. First, the state variable S_t is negatively correlated with the worker's productivity y_t. This follows from the negative sensitivity of S_t that we discuss in the previous section. Second, compensation c_t is overall smoother than output y_t, suggesting that the firm provides a significant amount of insurance to the worker. However, c_t is not smooth at all when S_t hits zero. Any increase in y_t results at this point in an equally large increase in c_t because of the binding quitting constraint.

The worker's career path starts at the quitting constraint. At $t = 0$, the worker receives the initial continuation value W_0 equal to her market value $V(y_0)$, i.e., $S_0 = 0$. The productivity shocks hitting the worker are initially favorable along the depicted sample path, i.e., S_t initially stays close to zero and the contract remains in the rigid-wage phase. In fact, we see that the worker initially receives frequent raises (increases in c_t needed to keep her from quitting), reflecting the fact that the upside risk to her productivity is uninsurable due to the possibility of quitting. Starting at $t = 0.20$, shocks become less favorable: output drops and S_t begins to increase. The contract remains in the rigid-wage phase until $t = 0.58$, where S_t crosses over S^* for the first time. Between the peak output time $t = 0.20$ and the crossing time $t = 0.58$ we observe constant compensation, which reflects full insurance of negative productivity shocks in the rigid-wage phase of the contract.

At $t = 0.58$, the contract exits the rigid-wage region and enters a pay-for-performance phase. Compensation becomes sensitive to performance in this phase, as market-based incentive are no longer sufficient to elicit effort. In addition, between $t = 0.58$ and the contact's re-entry into the rigid-wage region at $t = 2.16$ we can observe a negative trend in the path of compensation, which is consistent with the negative drift of c_t depicted in Figure 2a. In the second spell of the rigid-wage phase starting at $t = 2.16$, the path of compensation follows the same pattern as in the first spell starting at $t = 0$: positive shocks trigger compensation raises when $S_t = 0$, while negative shocks are fully absorbed by the firm.

The frequency with which the contract will make excursions into the pay-for-performance region, and the duration of these spells, depend on the parameters in the model. As we see in Figure 2b, the stationary (i.e., zero-drift) point of S_t is much lower than S^* in this parametrization. Pay-for-performance incentives, thus, will be needed rarely.[36] In contrast, under the parameter values of Figure 3, the drift of the state variable is everywhere positive, so the contract will rarely, if at all, return to the rigid-wage region after its first exit. We investigate the long-run properties of the optimal contract next.

[36] The sample path we present in this simulation is somewhat atypical. We chose it to highlight the spell of pay-for-performance incentives.

5.6 Market-based incentives in the long run

This section provides two results. The first result gives a sufficient condition for the existence of a stationary stochastic steady state (an invariant distribution) for the state variable S_t.

Theorem 2 *If in the model with full commitment the drift of the state variable S_t is negative, i.e., if $-\mu - a_h < 0$, then in the model with both frictions there exists an invariant distribution for the state variable S_t.*

This result is intuitive because a negative drift in S_t when S_t is large prevents S_t from diverging. A strictly positive drift in S_t at zero makes the lower bound a reflecting barrier for S_t. These two forces give rise to a non-degenerate stationary distribution in S_t in the long run.[37]

The second result uses the stationary distribution for S_t to examine the fraction of time that the optimal contract spends in the region with strong market-based incentives. Denote the invariant distribution of S_t by π.

Proposition 6 $\lim_{a_h \to \infty} \pi([S^*, \infty)) = 0$.

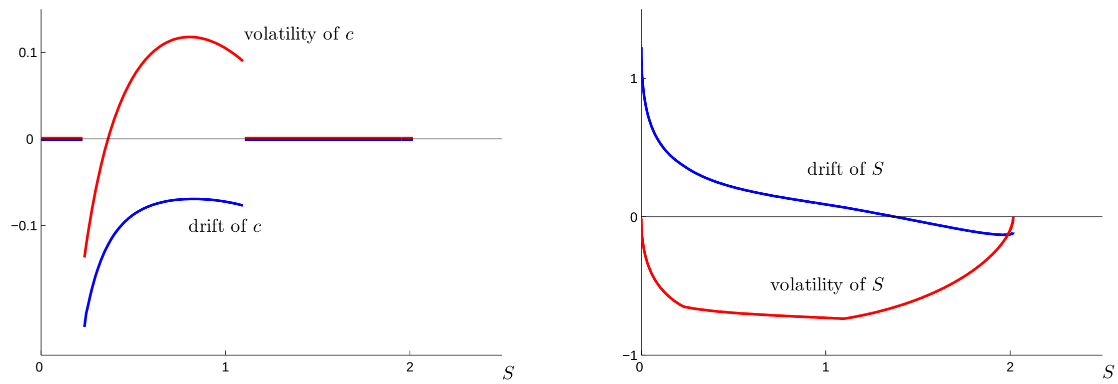
This proposition shows that if the worker's productivity has a sufficiently large drift under high effort, the optimal compensation contract will be free of pay-for-performance incentives most of the time. The argument for this result is that when a_h is large, the drift of the state variable is strongly negative at values of S_t strictly smaller than S^*. This makes events in which S_t leaves the region of strong market-based incentives very rare, and thus eliminates the need for pay-for-performance compensation incentives in equilibrium almost always.

6 Two extensions

In this paper, we adopt the CARA utility function, a Brownian motion productivity process, and one-sided limited commitment for the tractability of this framework. In particular, in this framework we can show that the high effort recommendation is optimal everywhere, and we can characterize the region of strong market-based incentives analytically. However, our main result showing that market-based incentives have a strong impact on optimal compensation contracts is not specific to the CARA-normal model with one-sided lack of commitment. In this section, we examine robustness of our result by considering two extensions. First, we consider two-sided lack of commitment (i.e., firms can fire workers) in the CARA-normal model. Second, we consider a model with log preferences and a geometric Brownian motion productivity process in the one-sided lack of commitment case. The cost of departing from the CARA-normal framework in this section is that we are only able to provide numerical solutions for these two extensions.[38]

[37] If $-\mu - a_h > 0$, we can show that $\lim_{t \to \infty} S_t = \infty$ with probability one.

[38] Within the CARA-normal framework with one-sided limited commitment, our analytical results can easily be extended to the case in which the absolute risk aversion parameter in the utility function is different from



(a) Dynamics of compensation. (b) Dynamics of S.

Figure 5: Example with two-sided lack of commitment and a large firing cost $F = 2.71$. The endogenous upper bound $\bar{S} = 2.02$.

6.1 Two-sided lack of commitment

Following Phelan (1995), we assume in this section that firms can fire workers upon incurring a deadweight cost $F \geq 0$. This introduces a participation constraint on the side of the firm: $J(S_t) \leq F$ at all t. This constraint implies that $S_t \leq \bar{S}$ at all t, where $\bar{S} = J^{-1}(F)$. Our model in Section 5 is a special case with $F = \infty$.

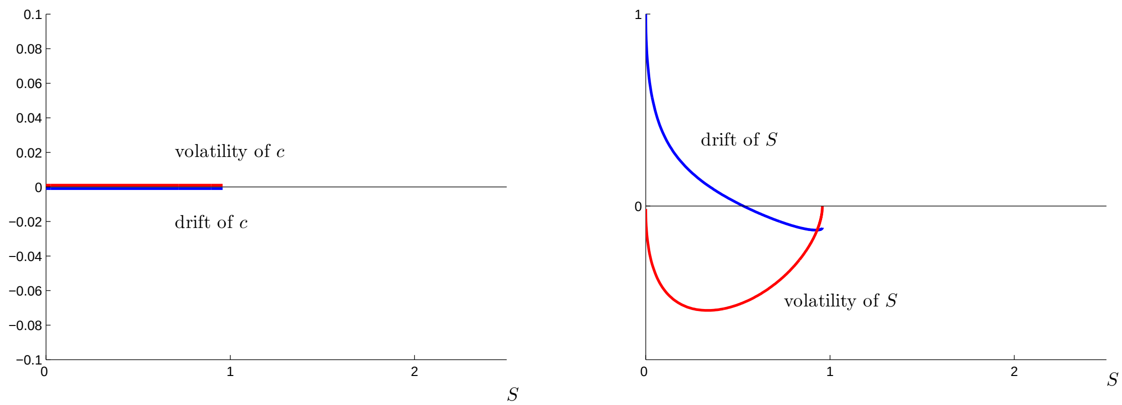
The numerical solutions we have obtained under various parametrizations show that market-based incentives become stronger when firm commitment becomes weaker. Figure 5 shows the equilibrium dynamics of compensation and the state variable when the firing cost F is set at 2.71.[39] With this F, the endogenous upper bound on S_t is determined at $\bar{S} = 2.02$. In $[0, \bar{S}]$, there are two regions with strong market-based incentives, where compensation is constant, and one region with weak market-based incentives, where pay-for-performance incentives are used.[40] In the lower region of constant compensation, as in the baseline model, the worker is motivated by the prospect of the raise that the firm must give her to keep her from quitting when S_t hits zero. In the upper region of constant compensation, the worker is motivated by the wage cut that she will have to accept in order to keep the firm from firing her when S_t reaches $\bar{S}$.

Like quitting, firing of workers never actually happens in equilibrium. Panel (b) of Figure 5 shows that when S_t approaches the firing boundary $\bar{S}$, the drift of S_t is negative and its sensitivity goes to zero, which means that at $\bar{S}$ the process S_t is reflected downward. Thus,

one. In this paper, we keep the absolute risk aversion parameter fixed at one because considering other values would make the notation less clear without adding any insight.

[39]Other parameters are the same as in Figure 3.

[40]Similar to the one-sided case, due to the non-separability of preferences between consumption and leisure, there is a discontinuity in the drift and in the sensitivity of compensation at the boundaries between the regions of strong and weak market-based incentives.



(a) Dynamics of compensation.

(b) Dynamics of S.

Figure 6: Example with two-sided lack of commitment and a small firing cost $F = 1.54$. The endogenous upper bound $\bar{S} = 0.96$.

zero and $\bar{S}$ are reflecting barriers that keep S_t in $[0, \bar{S}]$.

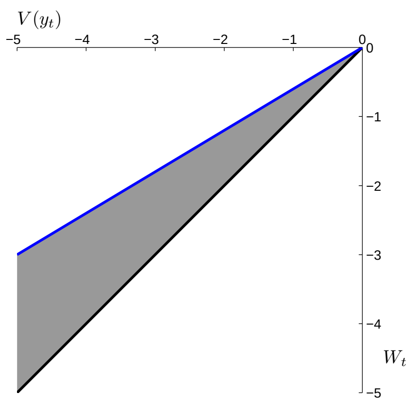
In addition to the example presented in Figure 5, we have computed examples with different levels of the firing cost F. In these examples, we have examined the structure of equilibrium compensation. When F decreases, $\bar{S} = J^{-1}(F)$ decreases, so the interval $[0, \bar{S}]$ shrinks. The middle region of that interval, where market-based incentives are weak, shrinks as well. In fact, the middle region shrinks faster than the interval $[0, \bar{S}]$.

For a small enough firing cost F, the region of weak market-based incentives vanishes completely and, hence, equilibrium compensation never uses pay-for-performance incentives. In these cases, compensation is piecewise constant: c_t is constant when S_t fluctuates inside the interval $(0, \bar{S})$, c_t increases when S_t hits zero, and c_t decreases when S_t hits $\bar{S}$. Compensation, therefore, has a "sticky wage" structure: small performance shocks do not affect the wage, but large shocks do. Figure 6 presents one such example. In this figure, $F = 1.54$ is smaller than in Figure 5.[41] As we see, the equilibrium firing threshold $\bar{S}$ is smaller here than in Figure 5, but remains positive, i.e., the firm still provides insurance to the worker. Panel (a) shows that the drift and the sensitivity of c_t are both zero everywhere inside $(0, \bar{S})$. As in the previous example, we can see in panel (b) that 0 and $\bar{S}$ are reflecting barriers for S_t.

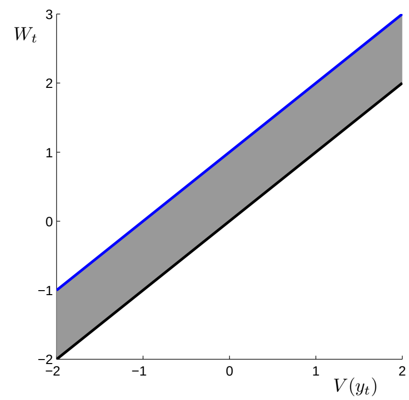
Also, we note in the above example that the IC constraint does not bind anywhere in $[0, \bar{S}]$. This means that the equilibrium contract is the same as what it would be if workers' effort were observable. With a small enough firing cost, therefore, the commitment friction becomes strong enough to completely crowd out the private information friction.

The lower the firing cost F, the less insurance firms provide in equilibrium. In the limiting case with $F = 0$, we have $\bar{S} = 0$ and firms provide no insurance, i.e., they simply pay workers the output workers produce: $c_t = y_t$ at all t.

[41] All other parameters are the same as in Figure 3 and Figure 5.



(a) CARA-normal.



(b) Log-geometric.

Figure 7: Regions of strong market-based incentives.

6.2 Log preferences and geometric Brownian motion

We have also studied numerically a version of our model with the log utility of consumption additively separable from the utility of leisure, with a geometric Brownian motion productivity process, and with one-sided commitment. In that framework, the high-effort action is not always optimal, but it is when slackness in the worker's quitting constraint is not too large. We have examined numerically the solution to the optimal contracting problem, and have found that strong market-based incentives also exist in this model.[42] Figure 7 shows the area of strong market-based incentives in our main CARA-normal model (panel (a)) and in a log-geometric model (panel (b)). The main conclusion of our previous analysis holds in the log-geometric framework: market-based incentives are strong when the quitting constraint is not very slack.

7 Testable predictions

Downward wage rigidity is a well-documented phenomenon. Consistent with other studies estimating the frequency of nominal wage changes, Gottschalk (2005) estimates the annual probability of a wage decline to be between 4 and 5 percent, and the probability of no wage change to be about 50 percent. That evidence would not be consistent with the career concerns model, the pure moral hazard model, or the rigid wage model, but it can be consistent with the predictions of our model. The career concerns model and the pure moral hazard model do not generate wage constancy, as wages change in these models as soon as new information carried by the worker's observed performance becomes available. The rigid wage model of Harris and Holmstrom (1982) does not predict any wage decreases. Our model can generate both wage constancy and wage decreases. In particular, the proportion of wage decreases can in our model

[42]Detailed solutions are available upon request.

be small relative to wage constancy or wage increases if the contract spends a large fraction of time in the rigid-wage region with occasional excursions into the pay-for-performance region.[43] As shown in Section 5.6, this pattern can be consistent with steady-state properties of the contract.

Our characterization of the equilibrium contract also provides testable predictions on the likelihood of the use of pay-for-performance compensation across occupations and over the life-cycle. Performance-based incentives should be more frequently observed a) in occupations in which workers do not acquire much general, transferable human capital but only firm-specific human capital, or none, b) when the growth of a worker's general productivity is slower, e.g., later in the life-cycle, c) when firing workers is costly, and d) when workers' past performance is harder for outsiders to observe. Gibbons and Murphy (1992), Loveman and O'Connell (1996), and Lazear (2000) provide evidence consistent with these predictions. In a rich data set on compensation of executives, Gayle et al. (ming) document that new hires' compensation is less closely tied to the firm's performance than the compensation of longer-tenured executives, again consistent with the predictions of our model.

8 Conclusion

In this paper, we build a model that lets us study contractual incentives jointly with external, market-based incentives similar to career concerns. In the model, external incentives arise out of a) the persistent impact of effort on the worker's productivity, and b) one-sided commitment. We show how external incentives change the structure of the optimal long-term contract: connecting pay to performance is only needed when performance is weak; it is not needed when performance is strong.

When we relax the assumption of full commitment on the side of the firm and allow for firing of workers upon paying a small firing cost, performance pay becomes completely unnecessary. If firms can fire workers, market-based incentives are stronger because workers are motivated not only by the prospect of a pay raise but also by the risk of being fired.

Our model is designed to study the impact of market-based incentives on the dynamics of compensation in situations where maximum effort is always desirable. Understanding the impact of market-based incentives on effort in addition to compensation is an interesting question for future research. In this paper, we abstract from search frictions in the labor market and from aggregate uncertainty. How they affect market-based incentives is another interesting question for future research.[44]

Our analysis suggests that market-based incentives exist in principal-agent relationships be-

[43] In the extension of our model studied in Section 6, wage declines can also be generated by a binding firing constraint for the firm.

[44] Rudanko (2011) studies long-term contracts in a search model with idiosyncratic and aggregate uncertainty under full information and full commitment. Cooley et al. (2013) study long-term contracts in a search model with limited commitment.

yond the particular setting of our model, as long as the agent's effort (or other action benefiting the principal) improves the agent's standing in the market outside the present principal-agent relationship. For this reason, we expect that market-based incentives play an important role in many firm-employee and, perhaps particularly so, firm-executive relationships. As well, market-based incentives may be important in lender-borrower relationships, where the borrower's outside option (e.g., refinancing terms) can depend on the performance of her outstanding debt.

Appendix

Proof of Proposition 1

We prove (13) first, (12) second, and (10) and (11) last. The proof of (13) proceeds in the following five steps: (i)-(v).

(i) Recall that (8) defines the firm's cost minimization problem at any time. In particular, the cost minimization problem at time 0 is

$$\begin{aligned} C(W, y) = \min_{(\boldsymbol{a}, \boldsymbol{c}) \in \Sigma(y)} \quad & \mathbb{E}^{\boldsymbol{a}} \left[\int_0^\infty r e^{-rt} (c_t - y_t) dt \right] \\ \text{subject to} \quad & \mathbb{E}^{\boldsymbol{a}} \left[\int_0^\infty r e^{-rt} U(c_t, a_t) dt \right] = W. \end{aligned} \tag{28}$$

The principle of optimality states that the solution to a time-0 dynamic programming problem also solves the problem starting from $t > 0$, i.e., $\mathbb{E}^{\boldsymbol{a}} \left[\int_0^\infty r e^{-rs} (c_{t+s} - y_{t+s}) ds | \mathcal{F}_t \right] = C(W_t, y_t)$. From the definition of equilibrium, we have $C(W, y) \geq 0$ for $W \geq V(y)$. This property and the quitting constraint $W_t \geq V(y_t)$ imply that $C(W_t, y_t) \geq 0$, or

$$\mathbb{E}^{\boldsymbol{a}} \left[\int_0^\infty r e^{-rs} (c_{t+s} - y_{t+s}) ds | \mathcal{F}_t \right] \geq 0, \forall t \geq 0. \tag{29}$$

(ii) We now define an auxiliary cost minimization problem in which the quitting constraint is replaced with the non-positive continuation profit constraint (29), and show that the cost function from this problem, $\tilde{C}(W, y)$, is the same as $C(W, y)$. For any y and $W \geq V(y)$, define $\tilde{C}(W, y)$ as

$$\begin{aligned} \tilde{C}(W, y) = \min_{(\boldsymbol{a}, \boldsymbol{c})} \quad & \mathbb{E}^{\boldsymbol{a}} \left[\int_0^\infty r e^{-rt} (c_t - y_t) dt \right] \\ \text{subject to} \quad & W_0 = W, (\boldsymbol{a}, \boldsymbol{c}) \text{ is incentive compatible, and (29),} \end{aligned} \tag{30}$$

where the process $\{y_t; t \geq 0\}$ starts from the initial condition $y_0 = y$. Since the solution to (28) is shown to satisfy (29) in step (i) of this proof, $C(W, y) \geq \tilde{C}(W, y) \geq 0$. If $W = V(y)$, then $C(V(y), y) = 0 = \tilde{C}(V(y), y)$. If $W > V(y)$, denote the contract solving (30) by

$(\boldsymbol{a}', \boldsymbol{c}')$. Define $\lambda \equiv \min\{t : W_t = V(y_t)\}$. Then a contract $(\boldsymbol{a}'', \boldsymbol{c}'')$ that is equal to $(\boldsymbol{a}', \boldsymbol{c}')$ on $[0, \lambda)$ but switches to the market contract after λ has the same cost as $(\boldsymbol{a}', \boldsymbol{c}')$, as both the market contract and the tail of $(\boldsymbol{a}', \boldsymbol{c}')$ have zero cost starting at λ. Since $(\boldsymbol{a}'', \boldsymbol{c}'')$ satisfies (3), it is feasible in (28). Hence $C(W, y) \leq C((\boldsymbol{a}'', \boldsymbol{c}'')) = C((\boldsymbol{a}', \boldsymbol{c}')) = \tilde{C}(W, y)$.

(iii) If a contract $(\boldsymbol{a}, \boldsymbol{c})$ delivers utility W, then $(\boldsymbol{a}, \boldsymbol{c} + x)$ delivers We^{-x} for any $x \in \mathbb{R}$. This is because $W = \mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}U(c_t, a_t)dt\right]$ if and only if

$$We^{-x} = e^{-x}\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}U(c_t, a_t)dt\right] = \mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}U(c_t + x, a_t)dt\right].$$

(iv) The incentive compatibility of $(\boldsymbol{a}, \boldsymbol{c})$ is equivalent to the incentive compatibility of $(\boldsymbol{a}, \boldsymbol{c} + x)$. In fact, the incentive compatibility of $(\boldsymbol{a}, \boldsymbol{c})$ requires that $\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}U(c_t, a_t)dt\right] \geq \mathbb{E}^{\boldsymbol{b}}\left[\int_0^\infty re^{-rt}U(c_t, b_t)dt\right]$ for any deviation strategy $\boldsymbol{b}$, which is equivalent to

$$\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}U(c_t + x, a_t)dt\right] \geq \mathbb{E}^{\boldsymbol{b}}\left[\int_0^\infty re^{-rt}U(c_t + x, b_t)dt\right].$$

(v) We now verify that $\tilde{C}(W, y) = \tilde{C}(We^y, 0)$. Suppose $(\boldsymbol{a}, \boldsymbol{c})$ solves the problem in $\tilde{C}(W, y)$. We verify that $(\boldsymbol{a}, \boldsymbol{c} - y)$ is feasible in the minimization problem defining $\tilde{C}(We^y, 0)$. First, steps (iii) and (iv) have shown that $(\boldsymbol{a}, \boldsymbol{c} - y)$ delivers utility We^y and is incentive compatible. Second, if $\boldsymbol{y} = \{y_t; t \geq 0\}$ starts with the initial condition $y_0 = y$ and $\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rs}(c_{t+s} - y_{t+s})ds|\mathcal{F}_t\right] \geq 0$, then $\boldsymbol{y}^0 = \{y_t^0; t \geq 0\}$ defined as $y_t^0 \equiv y_t - y \;\forall t$ starts with the initial condition $y_0^0 = 0$, and

$$\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rs}(c_{t+s} - y - y_{t+s}^0)ds|\mathcal{F}_t\right] = \mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rs}(c_{t+s} - y - (y_{t+s} - y))ds|\mathcal{F}_t\right] \geq 0.$$

Hence $(\boldsymbol{a}, \boldsymbol{c} - y)$ satisfies (29) in $\tilde{C}(We^y, 0)$ and so it is feasible in the problem defining $\tilde{C}(We^y, 0)$. Feasibility of $(\boldsymbol{a}, \boldsymbol{c} - y)$ in this problem implies that

$$\tilde{C}(We^y, 0) \leq \mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}(c_t - y - y_t^0)dt\right] = \mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}(c_t - y_t)dt\right] = \tilde{C}(W, y).$$

By a symmetric argument, we can show $\tilde{C}(We^y, 0) \geq \tilde{C}(W, y)$. Thus, $\tilde{C}(W, y) = \tilde{C}(We^y, 0)$. By step (ii), this implies (13).

We move on to showing (12). The proof is by contradiction. If $V(0)e^{-y} > V(y)$, then $0 = C(V(0), 0) = C(V(0)e^{-y}, y) > C(V(y), y) = 0$, which is a contradiction. If $V(0) < V(y)e^y$, then $0 = C(V(0), 0) < C(V(y)e^y, 0) = C(V(y), y) = 0$, which is again a contradiction.

Finally, we move on to proving (10) and (11). We need to show that if $(\boldsymbol{a}, \boldsymbol{c})$ is optimal in the contracting problem for $C(y, V(y))$ defined in (8), then $(\boldsymbol{a}, \boldsymbol{c} - y)$ is optimal in the contracting problem for $C(0, V(0))$. We first show that the candidate contract $(\boldsymbol{a}, \boldsymbol{c} - y)$ is feasible in this problem. Steps (iii) and (iv) above imply that this candidate contract is incentive compatible and delivers utility $V(y)e^y = V(0)$. The candidate contract satisfies the quitting constraint (3) because

$$\begin{aligned}\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rs}U(c_{t+s}-y,a_{t+s})ds\,|\mathcal{F}_t\right] &= \exp(y)\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rs}U(c_{t+s},a_{t+s})ds\,|\mathcal{F}_t\right] \\ &\geq \exp(y)V(y_t) \\ &= V(y_t-y) \\ &= V\left(y_t^0\right),\end{aligned}$$

where, as before, the income process y_t starts at y, and $y_t^0 \equiv y_t - y$ starts at 0. Thus, the candidate contract $(\boldsymbol{a}, \boldsymbol{c}-y)$ satisfies quitting, IC, and promise-keeping constraints, and so it is feasible in the contracting problem for $C(0, V(0))$.

To finish the proof, we show that the candidate contract $(\boldsymbol{a}, \boldsymbol{c}-y)$ attains $0 = C(0, V(0))$, and hence is optimal in this problem:

$$\begin{aligned}\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}((c_t-y)-y_t^0)dt\right] &= \mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}(c_t-y-(y_t-y))dt\right] \\ &= \mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}(c_t-y_t)dt\right] \\ &= 0.\end{aligned}$$

■

Proof of Proposition 2

The proof has three steps: the first step shows the binding IC constraint and (25), the second shows the optimality of high effort, and the third shows (24).

First, we show that

$$J_{FC}(S) = J_{FC}(0) + S, \text{ for all } S \in \mathbb{R}. \tag{31}$$

Indeed, if an IC contract $(\boldsymbol{a}, \boldsymbol{c})$ delivers to the worker initial utility $V_{FC}(0)$, then for any $S \in \mathbb{R}$ the contract $(\boldsymbol{a}, \boldsymbol{c}+S)$ is also IC and delivers to the worker initial utility $V_{FC}(0)\exp(-S) = V_{FC}(S)$. Hence, for any y, the principal's cost function under full commitment satisfies $C_{FC}(V_{FC}(S), y) = C_{FC}(V_{FC}(0), y) + S$. Setting $y = 0$ in this equality and using definition $J_{FC}(S) = C_{FC}(V_{FC}(S), 0)$, we obtain $J_{FC}(S) = J_{FC}(0) + S$.

Substituting (31) into the HJB equation (22) and using $J'_{FC} = 1$ and $J''_{FC} = 0$, we obtain

$$r\left(S_t + J_{FC}(0)\right) = rS_t - r\log(-V_{FC}(0)) + \min_{\hat{u}_t, \hat{\Delta}_t \geq -\hat{u}\beta}\left\{r(-\log(-\hat{u}_t)) + r\left(-1-\hat{u}_t\right) + \frac{1}{2}\hat{\Delta}_t^2 - a_h\right\}.$$

Canceling rS_t on both sides, we obtain a static minimization problem (controls do not change over time) determining the value of $J_{FC}(0)$

$$rJ_{FC}(0) = -r\log(-V_{FC}(0)) + \min_{\hat{u},\hat{\Delta}\geq -\hat{u}\beta}\left\{r(-\log(-\hat{u}_t)) + r\left(-1-\hat{u}\right) + \frac{1}{2}\hat{\Delta}^2 - a_h\right\}. \tag{32}$$

Since the expression under minimization is quadratic in $\hat{\Delta}$ and $-\hat{u}\beta > 0$, the IC constraint will bind and the optimal $\hat{\Delta} = -\hat{u}\beta$, which implies (25). The optimal $\hat{u}$ solves the convex problem

$$\min_{\hat{u}}\left\{-r\log(-\hat{u}) + r\left(-1-\hat{u}\right) + \frac{1}{2}\left(-\hat{u}\right)^2\beta^2\right\}.$$

The first-order condition of this problem is a quadratic equation in $(-\hat{u})$ given by

$$-1 + (-\hat{u}) + r^{-1}\beta^2\left(-\hat{u}\right)^2 = 0, \tag{33}$$

with a single positive root

$$-\hat{u} = \frac{\sqrt{1+4r^{-1}\beta^2}-1}{2r^{-1}\beta^2}.$$

This root is in Proposition 2 denoted by ρ. Because $0 < \frac{\sqrt{1+4x}-1}{2x} < 1$ for all $x > 0$, we have that $0 < \rho < 1$. Substituting $-\hat{u} = \rho$ and $\hat{\Delta} = \rho\beta$ into (32) yields

$$rJ_{FC}(0) = -r\log(-V_{FC}(0)) - r\log(\rho) + r\left(-1+\rho\right) + \frac{1}{2}\rho^2\beta^2 - a_h. \tag{34}$$

Second, we confirm that high effort is always optimal. Note that the value of $J_{FC}(0)$ under low effort would be determined by

$$rJ_{FC}(0) = -r\log(-V_{FC}(0)) + \min_{\hat{u},\hat{\Delta}\leq -\hat{u}\beta}\left\{r(-\log(-\hat{u})) + r\left(-1-\hat{u}_t\phi\right) + \frac{1}{2}\hat{\Delta}^2 - a_l\right\}, \tag{35}$$

where the optimal $\hat{\Delta} = 0$ and the optimal $\hat{u}$ solves $\min_{\hat{u}}\left\{-r\log(-\hat{u}) + r\left(-1-\hat{u}\phi\right)\right\}$, which has a unique solution $-\hat{u} = \phi^{-1}$. This implies that $J_{FC}(0)$ under low effort would be

$$rJ_{FC}(0) = -r\log(-V_{FC}(0)) - r\log(\phi^{-1}) - a_l.$$

Thus, high effort is optimal if and only if $-r\log(\phi^{-1}) - a_l \geq -r\log(\rho) + r\left(-1+\rho\right) + \frac{1}{2}\rho^2\beta^2 - a_h$. To prove this inequality, note that Assumption 1 implies $\beta < \sigma$ and

$$\begin{aligned} a_h - a_l - r\log(\phi^{-1}) \geq \frac{1}{2}\beta\sigma \geq \frac{1}{2}\beta^2 &= \left.\left(-r\log(-\hat{u}) + r\left(-1-\hat{u}\right) + \frac{1}{2}\left(-\hat{u}\right)^2\beta^2\right)\right|_{\hat{u}=-1} \\ &\geq -r\log(\rho) + r\left(-1+\rho\right) + \frac{1}{2}\rho^2\beta^2. \end{aligned}$$

Third, we show (24). From $u(c_t)/W_t = -\hat{u} = \rho$, we have $-\exp(-c_t) = W_t\rho$, which gives us that $dc_t = -d\log(-W_t) = d(S_t + y_t)$. Recalling (21), or using Ito's lemma again, we have

$$\begin{aligned} dc_t &= \left(r\left(-1-\hat{u}\right) + \frac{1}{2}\hat{\Delta}^2 \right) dt + \hat{\Delta} dz_t^a \\ &= -\left(r\left(1-\rho\right) - \frac{1}{2}\left(\rho\beta\right)^2 \right) dt + \rho\beta dz_t^a \\ &= -\mu dt + \rho\beta dz_t^a, \end{aligned}$$

where the second line uses optimal controls $-\hat{u} = \rho$ and $\hat{\Delta} = \rho\beta$, and the third line uses the definition of μ in Proposition 2. To see that $\mu > 0$ note that $r(1-\rho) - \frac{1}{2}(\rho\beta)^2 > r(1-\rho) - (\rho\beta)^2 = 0$, where the equality follows from (33). To obtain initial consumption c_0, note that $J_{FC}(0) = 0$ in equilibrium. This and (34) imply that

$$r\left(\log(-V_{FC}(0)) \;+ \log(\rho)\right) = r\left(-1+\rho\right) + \frac{1}{2}\rho^2\beta^2 - a_h = -\mu - a_h.$$

From $-\exp(-c_0) = W_0\rho = V_{FC}(y_0)\rho = V_{FC}(0)e^{-y_0}\rho$ we have

$$c_0 = y_0 - \left(\log(-V_{FC}(0)) \;+ \log(\rho)\right) = y_0 + \frac{\mu + a_h}{r}.$$

Solving $dc_t = -\mu dt + \rho\beta dz_t^a$ with this initial condition yields (24). ■

Proof of Proposition 3

We know from Grochulski and Zhang (2011, equations (11) and (12), page 2365) that the optimal compensation at time t is given by $c_t = u^{-1}(\bar{u}(m_t))$, where $\bar{u}$ is a strictly increasing function given by

$$\bar{u}(y) = V_{FI}(y) - \frac{V'_{FI}(y)}{\lim_{\varepsilon\downarrow 0} \frac{d}{d\varepsilon}\left(1 - \mathbb{E}_y^{a_h}\left[e^{-r\tau_{y+\varepsilon}}\right]\right)},$$

with $\tau_{y+\epsilon}$ denoting the hitting time of the level $y+\epsilon$. Because y_t is a Brownian motion with drift a_h, we know that $\mathbb{E}_y^{a_h}\left[e^{-r\tau_{y+\varepsilon}}\right] = e^{-\kappa\varepsilon}$, where κ is defined in Assumption 1.[45] Thus, $\lim_{\varepsilon\downarrow 0} \frac{d}{d\varepsilon}\left(1 - \mathbb{E}^{a_h}\left[e^{-r\tau_{y+\varepsilon}}\right]\right) = \kappa$ and the function $\bar{u}$ is simplifies to

$$\bar{u}(y) = V_{FI}(y) - \frac{-V_{FI}(y)}{\kappa} = \left(1 + \frac{1}{\kappa}\right) V_{FI}(y) = u(y)\left(1 + \frac{1}{\kappa}\right)(-V_{FI}(0)).$$

Therefore, optimal compensation satisfies $u(c_t) = u(m_t)\left(1 + \frac{1}{\kappa}\right)(-V_{FI}(0))$. Applying u^{-1} to both sides, we obtain $c_t = m_t - \log\left(1 + \frac{1}{\kappa}\right) - \log(-V_{FI}(0))$. From $J_{FI}(0) = 0$ we compute

[45] This expression is different in Grochulski and Zhang (2011), i.e., $\mathbb{E}_y^{a_h}\left[e^{-r\tau_{y+\varepsilon}}\right] = (\frac{y+\varepsilon}{y})^{-\kappa}$, because the income process considered there is a geometric Brownian motion.

$\log(-V_{FI}(0)) = \log\left(\frac{\kappa}{\kappa+1}\right) + \frac{\kappa\sigma^2}{2r}$, which gives us (26).

As in Grochulski and Zhang (2011), the worker's continuation value process satisfies

$$W_t = \left(1 - e^{-\kappa(m_t - y_t)}\right)\bar{u}(m_t) + e^{-\kappa(m_t-y_t)}V_{FI}(m_t) = \left(1 + \frac{1-e^{-\kappa(m_t-y_t)}}{\kappa}\right)V_{FI}(m_t),$$

from which we can compute the sensitivity of W_t as $-V_{FI}(m_t)e^{-\kappa(m_t-y_t)}\sigma$, which, with $u(c_t) = V_{FI}(m_t)\left(1+\frac{1}{\kappa}\right)$ gives us (27). ■

Proof of Corollary 1

The proof follows immediately from (27). We only need to check that $\delta > 0$, or $\frac{\kappa\sigma}{\kappa+1} > \beta$. Indeed,

$$\frac{\kappa\sigma}{\kappa+1} > \frac{r\sigma\log(\phi^{-1})}{a_h - a_l} > \frac{r\sigma(1-\phi)}{a_h - a_l} = \beta,$$

where the first inequality follows from Assumption 1. ■

Preliminary analysis of the HJB equation

Below, we will often use $\hat{u}$, $\hat{\Delta}$, J' and J'' as shorthand notation for $\hat{u}(S)$, $\hat{\Delta}(S)$, $J'(S)$ and $J''(S)$, respectively.

Lemma A.1 *The IC constraint is slack if and only if* $\frac{\sigma J'J''}{J'+J''} > \beta$. *When it is slack,*

$$\hat{\Delta} = \frac{\sigma J''}{J'+J''}, \tag{36}$$

$$\hat{u} = -J'^{-1}. \tag{37}$$

Proof The first-order conditions for $\hat{\Delta}$ and $\hat{u}$ are

$$\hat{\Delta} \geq \frac{\sigma J''}{J'+J''}, \quad \hat{u} \geq -J'^{-1},$$

with equalities if the IC constraint is slack. Thus, if the IC constraint is slack, then (36) and (37) hold, and $\hat{\Delta} > -\hat{u}\beta$ implies $\beta < \frac{\hat{\Delta}}{-\hat{u}} = \frac{\sigma J'J''}{J'+J''}$. If the IC constraint binds, then $\hat{\Delta} \geq \frac{\sigma J''}{J'+J''}$ and $\hat{u} \geq -J'^{-1}$, and thus $\hat{\Delta} = -\hat{u}\beta$ implies $\beta = \frac{\hat{\Delta}}{-\hat{u}} \geq \frac{\sigma J'J''}{J'+J''}$. ■

Let $\mathscr{H}(S, J', J'')$ denote the right-hand side of the HJB equation (22), that is

$$\mathscr{H}(S,J',J'') \equiv \min_{\hat{u},\hat{\Delta}\geq-\hat{u}\beta} \quad r(S - \log(-V(0)) - \log(-\hat{u})) + J'\left(r(-1-\hat{u}) + \frac{1}{2}\hat{\Delta}^2 - a_h\right) + \frac{1}{2}J''(\hat{\Delta}-\sigma)^2, \tag{38}$$

where J' and J'' are scalars. Whenever $\mathscr{H}(S, J', J'')$ is invertible in J'', we may rewrite the HJB equation as a second-order ordinary differential equation (ODE)

$$J''(S) = \mathscr{H}^{-1}(S, J, J'). \tag{39}$$

We study the invertibility of $\mathscr{H}(S, J', \cdot)$ next.

Lemma A.2 *If $J' \geq \frac{\kappa}{\kappa+1}$, then at any $J'' \in [0, \infty)$ the function $\mathscr{H}(S, J', J'')$ is strictly increasing in J'', and*

$$\hat{\Delta} < \sigma. \tag{40}$$

Proof The Envelope theorem states that $\frac{\partial \mathscr{H}}{\partial J''} = \frac{1}{2}(\hat{\Delta} - \sigma)^2$, which implies that $\mathscr{H}(S, J', J'')$ strictly increases in J'' whenever $\hat{\Delta} \neq \sigma$. It is then sufficient to show (40). Indeed, if the IC constraint is slack, then $\hat{\Delta} = \frac{\sigma J''}{J'+J''} < \sigma$. If the IC constraint binds, then

$$\hat{\Delta} = -\hat{u}\beta \leq \beta J'^{-1} \leq \frac{r(1-\phi)\sigma}{(a_h - a_l)\frac{\kappa}{\kappa+1}} < \frac{r(1-\phi)\sigma}{r\log(\phi^{-1})} < \sigma, \tag{41}$$

where the inequalities follow from $-\hat{u} \leq J'^{-1}$, $J' \geq \frac{\kappa}{\kappa+1}$, Assumption 1, and $\frac{1-\phi}{\log(\phi^{-1})} < 1$. ■

Because $\mathscr{H}$ is continuous and differentiable in (S, J', J''), so is the inverse function $\mathscr{H}^{-1}$. This implies that the second derivative J'' in (39) is continuous and differentiable, i.e., J''' exists.[46]

Lemma A.2 allows us to define the ODE (39) in the region

$$D \equiv \left\{(S, J, J') \in \mathbb{R}^3 : \mathscr{H}(S, J', 0) \leq rJ \leq \mathscr{H}(S, J', \infty) \text{ and } J' \geq \frac{\kappa}{\kappa+1}\right\}.$$

Next we derive an explicit functional form for $\mathscr{H}^{-1}(S, J, J')$ when the IC constraint is slack.

Lemma A.3 *If the IC constraint is slack, then*

$$J'' = \left(\frac{\sigma^2/2}{r(J - S + \log(-V(0)) - \log(J') - 1) + (r + a_h)J'} - \frac{1}{J'}\right)^{-1}. \tag{42}$$

Proof If the IC constraint is slack, substituting (36) and (37) into the HJB equation yields

$$rJ \;\; = \;\; rS - r\log(-V(0)) + r\log(J') + J'\left(r\left(-1 + \frac{1}{J'}\right) + \frac{1}{2}\left(\frac{\sigma J''}{J' + J''}\right)^2 - a_h\right)$$

[46] However, the fourth derivative J'''' may not exist: because the policy functions $(\hat{u}, \hat{\Delta})$ in (38) may have kinks at the boundaries between the regions of binding and slack IC constraints, $\frac{\partial(\mathscr{H}^{-1})}{\partial J} = 1/\frac{\partial \mathscr{H}}{\partial J''} = 2/(\hat{\Delta} - \sigma)^2$ may have kinks and the second derivative $\frac{\partial^2(\mathscr{H}^{-1})}{\partial J^2}$ may not exist.

$$+\frac{1}{2}J''\left(\frac{\sigma J''}{J'+J''}-\sigma\right)^2,$$

which simplifies to

$$rJ = rS - r\log(-V(0)) + r\log(J') - rJ' + r - a_h J' + \frac{1}{2}\sigma^2\frac{J'J''}{J'+J''}.$$

Solving for J'' in the above yields (42). ■

The next lemma studies the HJB equation at the boundary $S = 0$. Let $\alpha(S_t)$ and $\zeta(S_t)$ denote the drift and the sensitivity of S_t given in (21) evaluated at optimal controls $\hat{u}(S_t)$ and $\hat{\Delta}(S_t)$.

Lemma A.4 *In the model with two frictions,*

(i) $\frac{\kappa}{\kappa+1} \le J'(0) \le \frac{r}{r+a_h-\frac{1}{2}\sigma^2}$ *and* $0 \le \alpha(0) \le \frac{1}{2}(\kappa+1)\sigma^2$*,*

(ii) $J''(0) = \infty$ *and* $\zeta(0) = 0$*,*

(iii) the IC constraint is slack when the quitting constraint binds.

Proof That $\alpha(0) \ge 0$ and $\zeta(0) = 0$ follow from the nonnegativity of S_t at all t. In particular, $\zeta(0) \neq 0$ would imply $S_t < 0$ shortly after $S_t = 0$ because a typical Brownian motion sample path has infinite variation. From the law of motion (20) we have that $\alpha(0) = r\left(-1-\hat{u}(0)\right)+\frac{1}{2}\sigma^2-a_h$ and $\zeta(0) = \hat{\Delta}(0) - \sigma$.

(i) First, we show $J'(0) \ge \frac{\kappa}{\kappa+1}$. Note that $\frac{\kappa}{\kappa+1} = J'_{FI}(0)$ by part (i) of Lemma B.1. We are thus showing here that $J'(0) \ge J'_{FI}(0)$. By contradiction, suppose $J'_{FI}(0) > J'(0)$. Then

$$\begin{aligned}
& rJ_{FI}(0) + r\log(-V_{FI}(0)) \\
= \;& \min_{\hat{u}} r(-\log(-\hat{u})) + J'_{FI}(0)\left(r(-1-\hat{u}) - a_h + \frac{1}{2}\sigma^2\right) \\
= \;& r\log\left(\frac{\kappa+1}{\kappa}\right) + J'_{FI}(0)\left(r(-1+\frac{\kappa+1}{\kappa}) - a_h + \frac{1}{2}\sigma^2\right) \\
> \;& r\log\left(\frac{\kappa+1}{\kappa}\right) + J'(0)\left(r(-1+\frac{\kappa+1}{\kappa}) - a_h + \frac{1}{2}\sigma^2\right) \\
\ge \;& \min_{-\hat{u}\beta\le\sigma} r(-\log(-\hat{u})) + J'(0)\left(r(-1-\hat{u}) - a_h + \frac{1}{2}\sigma^2\right) \;= rJ(0) + r\log(-V(0)),
\end{aligned}$$

where the first inequality follows from $\left(r(-1+\frac{\kappa+1}{\kappa}) - a_h + \frac{1}{2}\sigma^2\right) > 0$, and the second inequality follows from $\frac{\kappa+1}{\kappa}\beta < \sigma$. Because $J_{FI}(0)$ is the minimum cost to deliver utility $V_{FI}(0)$ under one friction (limited commitment), the scalability in Proposition 1 implies that $J_{FI}(0) + \log(-V_{FI}(0)) - \log(-V(0))$ is the minimum cost to deliver utility $V(0)$ under one friction, which must be lower than $J(0)$, the cost to deliver the same utility $V(0)$ under two frictions. This contradicts the above inequality.

Second, since part (iii) shows that the IC constraint is slack, it follows from Lemma A.1 that $-\hat{u} = J'^{-1}$. Under this condition, $J'(0) \leq \frac{r}{r+a_h-\frac{1}{2}\sigma^2}$ is equivalent to $r\left(-1-\hat{u}(0)\right) + \frac{1}{2}\sigma^2 - a_h = \alpha(0) \geq 0$. Further, under $-\hat{u} = J'^{-1}$, $\frac{\kappa}{\kappa+1} \leq J'(0)$ is equivalent to $r\left(-1-\hat{u}(0)\right) + \frac{1}{2}\sigma^2 - a_h \leq r\left(-1+\frac{\kappa+1}{\kappa}\right) + \frac{1}{2}\sigma^2 - a_h = \frac{1}{2}(\kappa+1)\sigma^2$.

(ii) Suppose $J''(0) < \infty$ so that the assumptions of Lemma A.2 are met. But then (40) contradicts $\zeta(0) = \hat{\Delta}(0) - \sigma = 0$.

(iii) It follows from Assumption 1 and $J''(0) = \infty$ that $\beta = \frac{r\sigma(1-\phi)}{a_h-a_l} < \frac{r\sigma\log(\phi^{-1})}{a_h-a_l} < \sigma = \frac{\sigma J' J''}{J'+J''}$. By Lemma A.1, thus, the IC constraint is slack when $S_t = 0$.

■

Discussion. It is useful to briefly discuss the intuition behind Lemma A.4. As in the model with full information, the binding quitting constraint forces the firm to extinguish all volatility in S_t at the boundary $S_t = 0$. To achieve this, the sensitivity of the state variable has to be zero at the boundary, i.e., $\zeta(0) = 0$. This, in turn, is consistent with the firm's cost minimization if and only if the firm is infinitely averse to volatility in S_t at zero, hence $J''(0) = \infty$.

As in the full-information model, the firm provides no insurance to the worker when $S_t = 0$, as the continuation value inside the contract is at that point as volatile as the worker's market value, and, because providing insurance is feasible when $S_t > 0$, the firm induces a positive drift in S_t at $S_t = 0$. Comparing part (*i*) of Lemma A.4 with part (*i*) of Lemma B.1, however, we see that the firm's aversion to drift in the state variable, represented by the first derivative of the cost function, is larger in the two-friction model than in the full-information model. Accordingly, the positive drift in S_t at zero is smaller here than in the full-information model. This difference is due to the cost of future incentives. Part (*iii*) of Lemma A.4 shows that the IC constraint is slack when the quitting constraint binds. But we know from our analysis of the full-commitment model in Section 3 that the IC constraint binds when the quitting constraint is completely absent. Since the equilibrium contract in the two-friction model approximates the equilibrium contract of the full-commitment model when S_t is large, the IC constraint will bind in the two-friction model at S_t large enough. Inducing a positive drift in S_t in the two-friction model, therefore, has the disadvantage of making it more likely that the quitting constraint becomes sufficiently slack for the IC constraint to bind. This disadvantage is absent in the full-information model. The expected cost of future incentives, thus, makes positive drift in S_t more costly to the firm in the two-friction model, which is reflected in the firm's higher drift aversion $J'(0) \geq J'_{FI}(0)$ and lower drift of S_t at zero, as shown in part (*i*) of Lemma A.4.[47]

Closely related is the intuition for why the IC constraint is slack when the quitting constraint binds. Corollary 1 shows that the IC constraint is slack at $S_t = 0$ in the full-information model. This and the fact that the firm has a higher drift aversion in the two-friction model imply that

[47] Although conditions in part (*i*) of Lemma A.4 are given as weak inequalities, we show later that they are actually strict. Intuitively, the cost of future incentives is strictly positive because the IC constraint binds with strictly positive probability in equilibrium.

the IC constraint must also be slack at $S_t = 0$ in the two-friction model. Indeed, a smaller drift in S_t at zero implies that the worker in the two-friction model receives a higher utility flow $\hat{u}_t$. Since the sensitivity of S_t is zero at $S_t = 0$, the normalized, market-induced sensitivity of the worker's continuation value, $\hat{\Delta}_t$, must equal σ in both models. With higher $\hat{u}_t$ and the same $\hat{\Delta}_t$, the IC constraint is more slack in the two-friction model than in the full-information model.

Proof of Theorem 1

The proof is organized into three lemmas: Lemma A.5, Lemma A.9 and Lemma A.10. Three auxiliary lemmas are also proved: Lemma A.6, Lemma A.7 and Lemma A.8.

We start out by noting that because $J''(0) = \infty$, the HJB equation at $S = 0$ reduces to

$$J(0) = -\log(-V(0)) + \log(J'(0)) + 1 - J'(0)\frac{r + a_h - \frac{1}{2}\sigma^2}{r}.$$

Treating the right-hand side of this equation as a function of $J'(0)$, denote its value by $h(J'(0))$. Lemma A.4 now implies a range of possible values for $J(0)$, $J'(0)$ and $J''(0)$ given by $\big(J(0), J'(0), J''(0)\big) = (h(J'(0)), J'(0), \infty)$ for $J'(0) \in \left[\frac{\kappa}{\kappa+1}, \frac{r}{r+a_h-\frac{1}{2}\sigma^2}\right]$. Thus, the knowledge of $J'(0)$ would be sufficient to pinpoint the values for $J(0)$ and $J''(0)$. Not knowing $J'(0)$, however, we will proceed as follows. Denote by $K(S)$ the function solving the HJB equation starting from an initial condition $K'(0) \in \left[\frac{\kappa}{\kappa+1}, \frac{r}{r+a_h-\frac{1}{2}\sigma^2}\right]$. This gives us a set of candidate solution curves $K(S)$, one for each starting value $K'(0) \in \left[\frac{\kappa}{\kappa+1}, \frac{r}{r+a_h-\frac{1}{2}\sigma^2}\right]$. The true cost function J has to coincide with one of these curves. The asymptotic condition $\lim_{S\to\infty} J'(S) = 1 = J'_{FC}(S)$ will determine which of the candidate solution curves represents the true cost function J.

In order to carry out this program, we need to first show that the solution to the HJB equation (42) exists in the neighborhood of zero despite the fact that the HJB does not satisfy the Lipschitz condition at $S = 0$ (because $J''(0) = \infty$).

Lemma A.5 *The HJB equation has a unique candidate solution K in a neighborhood of $S = 0$ with the boundary condition $(K(0), K'(0), K''(0)) = (h(K'(0)), K'(0), \infty)$ for any $K'(0) \in \left(\frac{\kappa}{\kappa+1}, B\right)$, where*

$$B = \begin{cases} 1, & \text{if } a_h < \frac{1}{2}\sigma^2, \\ \frac{r}{r+a_h-\frac{1}{2}\sigma^2}, & \text{if } a_h \geq \frac{1}{2}\sigma^2. \end{cases}$$

The IC constraint is slack (i.e., $\frac{\sigma K' K''}{K'+K''} > \beta$) in a neighborhood of $S = 0$.

Proof Use a change of variable: define $x \equiv K'(S)$ and interpret both S and K as functions of

x. Since $\frac{dS}{dx} = \frac{1}{K''(S)}$ and $\frac{dK}{dx} = \frac{dK}{dS}\frac{dS}{dx} = \frac{x}{K''(S)}$, we have the differential equation system

$$\begin{aligned} \frac{dS}{dx} &= \frac{1}{2\sigma^{-2}\left(r(K - S + \log(-V(0)) - \log(x) - 1) + x\left(r + a_h\right)\right)} - \frac{1}{x}, \\ \frac{dK}{dx} &= \frac{x}{2\sigma^{-2}\left(r(K - S + \log(-V(0)) - \log(x) - 1) + x\left(r + a_h\right)\right)} - 1. \end{aligned}$$

The solution exists and is unique in a neighborhood of $(x, S, K) = (K'(0), 0, h(K'(0)))$ because the local Lipschitz condition is satisfied. When x is close to $K'(0)$, S and K both strictly increase in x because

$$\begin{aligned} \left.\frac{dS}{dx}\right|_{x=K'(0)} &= 0, \\ \left.\frac{dK}{dx}\right|_{x=K'(0)} &= x \left.\frac{dS}{dx}\right|_{x=K'(0)} = 0, \\ \left.\frac{d^2S}{dx^2}\right|_{x=K'(0)} &= \frac{2\sigma^{-2}\left(\frac{r}{x} - r - a_h\right)}{K'(0)^2} + \frac{1}{x^2} = \frac{2\sigma^{-2}\left(\frac{r}{K'(0)} - r - a_h + \frac{1}{2}\sigma^2\right)}{K'(0)^2} > 0, \\ \left.\frac{d^2K}{dx^2}\right|_{x=K'(0)} &= \left.\frac{dS}{dx}\right|_{x=K'(0)} + x \left.\frac{d^2S}{dx^2}\right|_{x=K'(0)} > 0. \end{aligned}$$

Part (iii) of Lemma A.4 shows that the IC constraint is slack at $S = 0$. Because $\frac{K'K''}{K'+K''} = \frac{x}{x\frac{1}{K''}+1} = \frac{x}{xS'(x)+1}$ is a continuous function of x, the IC constraint remains slack in a neighborhood of $x = K'(0)$. This neighborhood of x is mapped to a neighborhood of S because S is strictly increasing in x. ■

We can now move on to studying global properties of candidate solutions to the HJB equation. For a given candidate solution K, define

$$\bar{S} \equiv \min_{S}\left\{S > 0 : K'(S) = 1 \text{ or } K''(S) = 0 \text{ or } K''(S) = \infty\right\}, \tag{43}$$

with $\min \emptyset = \infty$.

The next three lemmas are auxiliary and will be used later.

Lemma A.6 *If $\bar{S} < \infty$ and $K''(\bar{S}) = 0$, then $K'(\bar{S}) < 1$.*

Proof By contradiction, suppose $K''(\bar{S}) = 0$ and $K'(\bar{S}) = 1$. Then the function $K(\cdot)$ that satisfies $K(S) = K(\bar{S}) + S - \bar{S}$ for all S solves the HJB equation. This violates the condition that $K''(0) = \infty$. ■

Lemma A.7 *If K is a candidate solution with $\bar{S} < \infty$ and $K''(\bar{S}) = \infty$, then $r(-1 - \hat{u}(\bar{S})) + \frac{1}{2}\sigma^2 - a_h < 0$.*

Proof By contradiction, suppose $r(-1-\hat{u}(\bar{S}))+\frac{1}{2}\sigma^2-a_h \geq 0$. The HJB equation at $S=\bar{S}$ is

$$rK(\bar{S}) \;=\; r(\bar{S}-\log(-V(0))-\log(-\hat{u}(\bar{S}))) + K'(\bar{S})\left(r(-1-\hat{u}(\bar{S}))+\frac{1}{2}\sigma^2-a_h\right).$$

Because $r(-1-\hat{u}(\bar{S}))+\frac{1}{2}\sigma^2-a_h \geq 0$ and $K'(\bar{S})>K'(0)$,

$$\begin{aligned} rK(\bar{S}) &\geq r(\bar{S}-\log(-V(0))-\log(-\hat{u}(\bar{S}))) + K'(0)\left(r(-1-\hat{u}(\bar{S}))+\frac{1}{2}\sigma^2-a_h\right) \\ &\geq r\bar{S}+\min_{\hat{u}} r(-\log(-V(0))-\log(-\hat{u})) + K'(0)\left(r(-1-\hat{u})+\frac{1}{2}\sigma^2-a_h\right) \\ &= r\bar{S}+rK(0), \end{aligned}$$

where the equality follows from the HJB equation at $S=0$. This contradicts the fact that $K'(S)<1$ for all $S\in[0,\bar{S}]$. ■

Lemma A.8 *If K is a candidate solution with $\bar{S}=\infty$, then $\lim_{S\to\infty}K'(S)=1$.*

Proof Suppose by contradiction $G\equiv\lim_{S\to\infty}K'(S)\neq 1$. Since $K'(S)<1$ for all S, $G<1$. Then

$$\begin{aligned} 0 &> rK(S)-r(K(0)+GS) \\ &= \min_{\hat{u},\hat{\Delta}}\left\{r(S-\log(-V(0))-\log(-\hat{u}))+K'(S)\left(r(-1-\hat{u})+\frac{1}{2}\hat{\Delta}^2-a_h\right)\right. \\ &\quad \left.+\frac{1}{2}K''(S)(\hat{\Delta}-\sigma)^2\right\}-r(K(0)+GS) \\ &\geq r(1-G)S+\min_{\hat{u}} r(-\log(-V(0))-\log(-\hat{u}))+K'(S)\left(r(-1-\hat{u})-a_h\right)-rK(0) \\ &\to \infty,\ \text{as } S\to\infty. \end{aligned}$$

This is a contradiction. ■

We now move on to two key lemmas of this proof.

Lemma A.9 *There exists a unique $K'(0)\in(\frac{\kappa}{\kappa+1},B)$ such that the candidate solution K satisfies $\bar{S}=\infty$.*

Proof **Existence:** Suppose by contradiction that all candidate solutions have $\bar{S}<\infty$. The rest of the proof proceeds in several steps.

(i) The solution curves starting with different $K'(0)$ are ordered: higher $K'(0)$ leads to permanently higher solution curves. Suppose there are two curves K_1 and K_2 with initial conditions $K_1(0)<K_2(0)$ and $K_1'(0)<K_2'(0)$, then $K_1'(S)<K_2'(S)$ for all $S\in[0,\min\{\bar{S}_1,\bar{S}_2\}]$. If not, define $\underline{S}\equiv\min\{S: K_1'(S)=K_2'(S)\}$. Because $K_1'(S)<K_2'(S)$ for all $S\leq\underline{S}$, $K_1(\underline{S})<K_2(\underline{S})$. Hence the HJB equation and $K_1'(\underline{S})=K_2'(\underline{S})$ imply

that $K_1''(\underline{S}) < K_2''(\underline{S})$, which means that $K_1'(S) > K_2'(S)$ when $\underline{S} - S > 0$ is small. This contradicts the definition of $\underline{S}$.

(ii) Define

$$\begin{aligned} U &\equiv \{K'(0) : \bar{S} < \infty, \text{ either } K'(\bar{S}) = 1 \text{ or } K''(\bar{S}) = \infty\}, \\ L &\equiv \{K'(0) : \bar{S} < \infty, K''(\bar{S}) = 0\}. \end{aligned}$$

It follows from Lemma A.6 that $U \cap L = \emptyset$. We show below that both U and L are nonempty and open, which generates a contradiction because $(\frac{\kappa}{\kappa+1}, B) = U \cup L$ is a connected set.

(iii) U is open. Take a $K'(0) \in U$. We will show that there exists a $\delta > 0$ such that if $|K_1'(0) - K'(0)| \leq \delta$, then $K_1'(0) \in U$. Since $K'(0) \in U$, $\bar{S} < \infty$. Two cases need to be considered: $K'(\bar{S}) = 1$, and $K''(\bar{S}) = \infty$. In the first case, because $K''(\bar{S}) > 0$, there exists a small $\epsilon > 0$ such that $K'(\bar{S} + \epsilon) > 1$. Because the solution of a differential equation depends continuously on its initial condition, there exists a small $\delta > 0$, such if $|K_1'(0) - K'(0)| \leq \delta$, then

$$K_1'(\bar{S} + \epsilon) > 1, \tag{44}$$

$$\sup_{S \in [0, \bar{S} + \epsilon]} \left| \frac{1}{K_1''(S)} \right| < \infty. \tag{45}$$

Inequality (44) implies $\bar{S}_1 < \bar{S} + \epsilon$. It follows from (45) that $K_1''(\bar{S}_1) > 0$, hence $K_1'(0) \notin L$. Thus, $K_1'(0) \in U$.

In the second case, recall that the HJB equation is solved by a change of variable whenever $K''(S) = \infty$. Then

$$\left.\frac{dS}{dx}\right|_{x=K'(\bar{S})} = 0, \quad \left.\frac{d^2 S}{dx^2}\right|_{x=K'(\bar{S})} = \frac{2\sigma^{-2}\left(r(-1-\hat{u}) - a_h + \frac{1}{2}\sigma^2\right)}{x^2} < 0,$$

where the inequality follows from Lemma A.7. Hence there exists a small $\epsilon > 0$ such that $\frac{dS}{dx}|_{x=K'(\bar{S})+\epsilon} < 0$. Because the solution of a differential equation depends continuously on its initial condition, there exists a small $\delta > 0$ such that if $|K_1'(0) - K'(0)| \leq \delta$, then

$$\left.\frac{dS_1}{dx}\right|_{x=K'(\bar{S})+\epsilon} < 0, \tag{46}$$

$$\sup_{S \in [0, S_1(K'(\bar{S})+\epsilon)]} \left| \frac{1}{K_1''(S)} \right| = \sup_{x \in [K_1'(0), K'(\bar{S})+\epsilon]} \left| \frac{dS_1}{dx} \right| < \infty. \tag{47}$$

Inequality (46) implies $\bar{S}_1 < S_1(K'(\bar{S}) + \epsilon)$. It follows from (47) that $K_1''(\bar{S}_1) > 0$, hence $K_1'(0) \notin L$. Thus, $K_1'(0) \in U$.

(iv) L is open. Recall from Lemma A.6 that if $K'(0) \in L$, then $K''(\bar{S}) = 0$ and $K'(\bar{S}) < 1$. Differentiating the HJB equation and applying the Envelope theorem yield[48]

$$0 = r + K'' \left(r(-1-\hat{u}) + \frac{1}{2}\hat{\Delta}^2 - a_h \right) + \frac{1}{2}K'''(\sigma - \hat{\Delta})^2 - rK'.$$

Hence $K''(\bar{S}) = 0$ and $K'(\bar{S}) < 1$ imply

$$\frac{1}{2}K'''(\bar{S})(\sigma - \hat{\Delta})^2 = r(K'-1) - K'' \left(r(-1-\hat{u}) + \frac{1}{2}\hat{\Delta}^2 - a_h \right) < 0.$$

Therefore $K'''(\bar{S}) < 0$ and there exists a small $\epsilon > 0$ such that $K''(\bar{S}+\epsilon) < 0$.

Pick a small $\epsilon_1 > 0$, such that $K'(\epsilon_1)$ satisfies $r(-1 + \frac{1}{K'(\epsilon_1)}) + \frac{1}{2}\sigma^2 - a_h > 0$. Recall that the HJB equation is solved in a neighborhood of $S = 0$ by a change of variable. For convenience, we denote the solution for an initial condition $K_1'(0)$ by $S_1(x)$ when $x \in [K'(0), K'(\epsilon_1)]$. Because the solution of a differential equation depends continuously on its initial condition, there exists a small $\delta > 0$, such that if $|K_1'(0) - K'(0)| \leq \delta$, then

$$K_1''(\bar{S}+\epsilon) \quad < \quad 0, \tag{48}$$

$$\sup_{S \in [0, \bar{S}+\epsilon]} \left| K_1'(S) \right| \quad < \quad 1, \tag{49}$$

$$\sup_{S \in [S_1(K'(\epsilon_1)), \bar{S}+\epsilon]} \left| K_1''(S) \right| \quad < \quad \infty. \tag{50}$$

Inequality (48) implies $\bar{S}_1 < \bar{S} + \epsilon$. If $\bar{S}_1 \in (0, S_1(K'(\epsilon_1))]$ (i.e., $K_1'(\bar{S}_1) \leq K'(\epsilon_1)$), because $r(-1 + \frac{1}{K_1'(\bar{S}_1)}) + \frac{1}{2}\sigma^2 - a_h > 0$, Lemma A.7 implies that $K_1''(\bar{S}_1) < \infty$. If $\bar{S}_1 \in [S_1(K'(\epsilon_1)), \bar{S}+\epsilon]$, (50) implies that $K_1''(\bar{S}_1) < \infty$. It follows from (49) and $K_1''(\bar{S}_1) < \infty$ that $K_1'(0) \notin U$. Hence $K_1'(0) \in L$ if $|K_1'(0) - K'(0)| \leq \delta$.

(v) $L \neq \emptyset$. We will show that $\frac{\kappa}{\kappa+1} \in L$. By contradiction, suppose $\frac{\kappa}{\kappa+1} \in U$. That is, if $K'(0) = \frac{\kappa}{\kappa+1}$, then either $K'(\bar{S}) = 1$ or $K''(\bar{S}) = \infty$. The HJB equations for J_{FI} and K imply that if $J_{FI}(S) + \log(-V_{FI}(0)) \geq K(S) + \log(-V(0))$ and $J_{FI}'(S) = K'(S)$, then $J_{FI}''(S) \geq K''(S)$. Hence, the same argument as in part (i) shows that $J_{FI}'(S) \geq K'(S)$ for all $S \leq \bar{S}$. It follows from $J_{FI}'(S) < 1, \forall S$ that $K'(\bar{S}) < 1$ and $K''(\bar{S}) = \infty$. A contradiction arises as follows.

$$\begin{aligned}
& rJ_{FI}(\bar{S}) + r\log(-V_{FI}(0)) \\
= \quad & \min_{\hat{u},\hat{\Delta}} r(\bar{S} - \log(-\hat{u})) + J_{FI}'(\bar{S}) \left(r(-1-\hat{u}) + \frac{1}{2}\hat{\Delta}^2 - a_h \right) + \frac{1}{2}J_{FI}''(\bar{S})(\hat{\Delta} - \sigma)^2 \\
< \quad & \min_{\hat{u}} r(\bar{S} - \log(-\hat{u})) + J_{FI}'(\bar{S}) \left(r(-1-\hat{u}) + \frac{1}{2}\sigma^2 - a_h \right)
\end{aligned}$$

[48] The third derivative K''' exists because $K''(S) = \mathscr{H}^{-1}(S, K, K')$ and $\mathscr{H}^{-1}$ is differentiable.

$$\leq \quad r(\bar{S} - \log(-\hat{u}(\bar{S}))) + J'_{FI}(\bar{S})\left(r(-1-\hat{u}(\bar{S})) + \frac{1}{2}\sigma^2 - a_h\right),$$

where $\hat{u}(\bar{S}) = -(K'(\bar{S}))^{-1}$ is the optimal $\hat{u}$ at $\bar{S}$ for $K(\bar{S})$. Because $J'_{FI}(\bar{S}) \geq K'(\bar{S})$ and $r(-1-\hat{u}(\bar{S})) + \frac{1}{2}\sigma^2 - a_h < 0$ (shown by Lemma A.7),

$$\begin{aligned} & r(\bar{S} - \log(-\hat{u}(\bar{S}))) + J'_{FI}(\bar{S})\left(r(-1-\hat{u}(\bar{S})) + \frac{1}{2}\sigma^2 - a_h\right) \\ \leq \quad & r(\bar{S} - \log(-\hat{u}(\bar{S}))) + K'(\bar{S})\left(r(-1-\hat{u}(\bar{S})) + \frac{1}{2}\sigma^2 - a_h\right) = rK(\bar{S}) + r\log(-V(0)), \end{aligned}$$

which is a contradiction as $J_{FI}(0) + \log(-V_{FI}(0)) = K(0) + \log(-V(0))$ and $J'_{FI}(S) \geq K'(S)$ imply $J_{FI}(\bar{S}) + \log(-V_{FI}(0)) \geq K(\bar{S}) + \log(-V(0))$.

(vi) $U \neq \emptyset$. First, suppose $a_h < \frac{1}{2}\sigma^2$. If $1 - K'(0) > 0$ is sufficiently small, then $K(\cdot)$ will reach $K' = 1$. Second, suppose $a_h \geq \frac{1}{2}\sigma^2$. If $K'(0) = B$, then we show that $\frac{dS}{dx}|_{x=B+\epsilon} < 0$ for small $\epsilon > 0$. To prove this, note that, similar to the proof in Lemma A.5,

$$\begin{aligned} \left.\frac{dS}{dx}\right|_{x=K'(0)} &= 0 = \left.\frac{dK}{dx}\right|_{x=K'(0)}, \\ \left.\frac{d^2S}{dx^2}\right|_{x=K'(0)} &= \frac{2\sigma^{-2}\left(\frac{r}{K'(0)} - r - a_h + \frac{1}{2}\sigma^2\right)}{K'(0)^2} = 0, \\ \left.\frac{d^2K}{dx^2}\right|_{x=K'(0)} &= \left.\frac{dS}{dx}\right|_{x=K'(0)} + x\left.\frac{d^2S}{dx^2}\right|_{x=K'(0)} = 0. \end{aligned}$$

The Taylor expansion of $2\sigma^{-2}\left(r(K - S + \log(-V(0)) - \log(x) - 1) + x\,(r + a_h)\right)$ is

$$\begin{aligned} & K'(0) + 2\sigma^{-2}\left(r\left(\frac{dK}{dx} - \frac{dS}{dx} - \frac{1}{x}\right) + (r + a_h)\right)(x - K'(0)) \\ & +2\sigma^{-2}r\left(\frac{d^2K}{dx^2} - \frac{d^2S}{dx^2} + \frac{1}{x^2}\right)(x - K'(0))^2 + o((x - K'(0))^2) \\ = \quad & x + \frac{2\sigma^{-2}r}{(K'(0))^2}(x - K'(0))^2 + o((x - K'(0))^2) > x, \end{aligned}$$

where the inequality holds when $x - K'(0) > 0$ is small, since $\frac{2\sigma^{-2}r}{(K'(0))^2} > 0$. Therefore,

$$\left.\frac{dS}{dx}\right|_{x=B+\epsilon} = \frac{1}{2\sigma^{-2}\left(r(K - S + \log(-V(0)) - \log(x) - 1) + x\,(r + a_h)\right)} - \frac{1}{x} < 0,$$

for small $\epsilon > 0$. Because the solution of a differential equation depends continuously on its initial condition, there exists a small $\delta > 0$, such that if the initial condition

$K_1'(0) \in (B-\delta, B)$, then

$$\left.\frac{dS_1}{dx}\right|_{x=B+\epsilon} < 0, \tag{51}$$

$$\sup_{S\in[0,S_1(B+\epsilon)]}\left|\frac{1}{K_1''(S)}\right| = \sup_{x\in[K_1'(0),B+\epsilon]}\left|\frac{dS_1}{dx}\right| < \infty. \tag{52}$$

It follows from $\frac{dS_1}{dx}|_{x=K_1'(0)} > 0$ and (51) that $\frac{dS_1}{dx} = 0$ for some $\hat{S} \in (0, S_1(B+\epsilon))$. Because $\frac{dS_1}{dx} = \frac{1}{K_1''(S)}$, we know that $K_1''(\hat{S}) = \infty$. Hence $\bar{S}_1 \le \hat{S}$ must be finite. It follows from (52) that $K_1''(\bar{S}_1) > 0$, hence $K_1'(0) \notin L$ and $K_1'(0) \in U$.

Uniqueness: By contradiction, suppose there are two initial conditions $K_1'(0) < K_2'(0)$ with $\bar{S}_1 = \bar{S}_2 = \infty$. Subtracting one HJB equation from the other yields

$$\begin{aligned} & r(K_2(S) - K_1(S)) \\ = & \min_{\hat{u},\hat{\Delta}}\left\{-r\log(-\hat{u}) + K_2'(S)\left(r(-1-\hat{u}) + \frac{1}{2}\hat{\Delta}^2 - a_h\right) + \frac{1}{2}K_2''(S)(\hat{\Delta}-\sigma)^2\right\} \\ & -\min_{\hat{u},\hat{\Delta}}\left\{-r\log(-\hat{u}) + K_1'(S)\left(r(-1-\hat{u}) + \frac{1}{2}\hat{\Delta}^2 - a_h\right) + \frac{1}{2}K_1''(S)(\hat{\Delta}-\sigma)^2\right\}. \end{aligned}$$

The left-hand side is positive at $S = 0$ and is strictly increasing with S, as shown in part (i) of the proof of existence. Lemma A.8 implies that $\lim_{S\to\infty} K_1'(S) = \lim_{S\to\infty} K_2'(S) = 1$. For any $\epsilon > 0$, there exists a large S such that $0 < K_1''(S) + K_2''(S) < \epsilon$. Therefore, the right-hand side can be made as small as needed if S is large. This is a contradiction. ■

Lemma A.10 *The candidate solution K with $\bar{S} = \infty$ is the true cost function J.*

Proof Because the technique of using the HJB equation to verify the optimality of K is standard, we omit the details of the steps involved. We verify two things:

(i) The cost of any IC contract is weakly higher than $K(S)$.

(ii) There exists an IC contract whose cost equals $K(S)$.

To see (i), pick an IC contract starting at $S_0 = S \ge 0$ and consider the stochastic process $\{S_t; t \ge 0\}$ in this contract. Define

$$M_t \equiv \int_0^t (c_s - y_s) r e^{-rs} ds + e^{-rt} K(S_t). \tag{53}$$

The HJB equation implies that M_t is a submartingale (i.e., it has a nonnegative drift), hence

$$K(S) = M_0 \le \mathbb{E}[M_\infty] = \mathbb{E}\left[\int_0^\infty (c_s - y_s) r e^{-rs} ds\right]. \tag{54}$$

To see (ii), construct a stochastic process $\{S_t; t \geq 0\}$ using $S_0 = S$ and the policy functions implied by the HJB equation for K. Denote the contract generated by $\{S_t; t \geq 0\}$ and the policy functions as σ^*. Then M_t defined in (53) is a martingale, and the inequality in (54) is replaced with an equality. This shows that the cost of σ^* is $K(S)$. ■

Proof of Proposition 4

We will show the existence of a unique $S^* > 0$ such that

$$\frac{\sigma J' J''}{J' + J''} \quad \begin{cases} > \beta, & \text{if } S < S^*, \\ = \beta, & \text{if } S = S^*, \\ < \beta, & \text{if } S > S^*. \end{cases}$$

By Lemma A.1, this will show that the IC constraint is slack if and only if $S_t < S^*$.
Existence of S^*: Lemma A.5 shows that $\frac{\sigma J'(S) J''(S)}{J'(S)+J''(S)} > \beta$ when S is small. If S^* does not exist, then $\frac{\sigma J'(S) J''(S)}{J'(S)+J''(S)} > \beta$ for all S. Recall the definition of $\bar{S}$ in (43) and recall that J satisfies $\bar{S} = \infty$ by Lemma A.9. The definition of $\bar{S}$ implies that $J''(S) > 0$ for all $S > 0$. This further implies $1 > \frac{J'(S)}{J'(S)+J''(S)}$ and

$$\sigma J''(S) > \frac{\sigma J'(S) J''(S)}{J'(S) + J''(S)} \geq \beta, \text{ for all } S,$$

which contradicts the fact that $J'(S) < 1$ for all S.
Uniqueness of S^*: It is sufficient to show that if $\frac{\sigma J'(S^*) J''(S^*)}{J'(S^*)+J''(S^*)} = \beta$ for some S^*, then $\frac{\sigma J'(S) J''(S)}{J'(S)+J''(S)} > \beta$ for $S < S^*$. There are two steps: the first step shows a property (55), which is used in the second step to complete the proof of uniqueness.

First, we show that if $\frac{\sigma J' J''}{J'+J''} \geq \beta$, then

$$rJ' + a_h \frac{J' J''}{J' + J''} < r. \tag{55}$$

If $a_h \leq 0$, then (55) is obvious because $J' < 1$. When $a_h > 0$, by contradiction, suppose that $\frac{\sigma J' J''}{J'+J''} \geq \beta$ and

$$rJ' + a_h \frac{J' J''}{J' + J''} \geq r \tag{56}$$

at some $\hat{S}$. Starting from $\hat{S}$, solve the differential equation

$$rJ = r\left(S - \log(-V(0)) + \log(J')\right) + J'\left(-r - a_h\right) + r + \frac{\sigma^2}{2} \frac{J' J''}{J' + J''}. \tag{57}$$

Differentiating (57) with respect to S yields

$$rJ' = r\left(1+\frac{J''}{J'}\right) + J''\left(-r-a_h\right) + \frac{\sigma^2}{2}\frac{d\left(\frac{J'J''}{J'+J''}\right)}{dS},$$

which implies

$$\begin{aligned}\frac{\sigma^2}{2}\frac{d\left(\frac{J'J''}{J'+J''}\right)}{dS} &= rJ' - r\left(1+\frac{J''}{J'}\right) + J''\left(r+a_h\right)\\ &= \frac{J'+J''}{J'}\left(rJ' + a_h\frac{J'J''}{J'+J''} - r\right) \geq 0, \qquad (58)\end{aligned}$$

where the inequality uses (56). Hence we have that either $\frac{d\left(\frac{J'J''}{J'+J''}\right)}{dS} > 0$, or $\frac{d\left(\frac{J'J''}{J'+J''}\right)}{dS} = 0$. In the latter case, $rJ' + a_h\frac{J'J''}{J'+J''} - r = 0$ and differentiating (58) yields

$$\begin{aligned}\frac{\sigma^2}{2}\frac{d^2\left(\frac{J'J''}{J'+J''}\right)}{dS^2} &= \frac{d\left(\frac{J'+J''}{J'}\right)}{dS}\left(rJ' + a_h\frac{J'J''}{J'+J''} - r\right) + \frac{J'+J''}{J'}\left(rJ'' + a_h\frac{d\left(\frac{J'J''}{J'+J''}\right)}{dS}\right)\\ &= 0 + \frac{J'+J''}{J'}J'' > 0.\end{aligned}$$

In both cases, there exists a small $\epsilon > 0$ such that $\frac{J'J''}{J'+J''}$ is strictly increasing in $[\hat{S}, \hat{S}+\epsilon]$. Hence on $[\hat{S}, \hat{S}+\epsilon]$ the solution J to (57) does satisfy the HJB equation and the IC constraint is slack. If we extend J beyond $\hat{S}+\epsilon$, $\frac{J'J''}{J'+J''}$ is always strictly increasing, because J' is increasing and a_h is positive in (58). Hence,

$$J'' > \frac{J'J''}{J'+J''} > \frac{J'(\hat{S})J''(\hat{S})}{J'(\hat{S})+J''(\hat{S})}, \text{ for all } S > \hat{S},$$

contradicting the fact that $J'(S) < 1$ for all S.

Second, we show that $\frac{\sigma J'(S)J''(S)}{J'(S)+J''(S)} > \beta$ for all $S < S^*$. Solve the differential equation (57) backward on $[0, S^*]$. Equations (55) and (58) show that $\frac{J'(S)J''(S)}{J'(S)+J''(S)}$ is strictly decreasing in S. Hence the solution J to (57) does satisfy the HJB equation and the IC constraint is slack.

This completes the proof of the first statement in Proposition 4. To prove the second statement, we now show that both the drift and the sensitivity of compensation are zero when $S \leq S^*$.

It follows from $\hat{u} = \frac{u(c)}{-W}$ and $S = \log(\frac{V(y)}{W})$ that

$$c = -\log(-\hat{u}) - \log(-W) = -\log(-\hat{u}) + S + y - \log(-V(0)).$$

If $S \leq S^*$, then $-\hat{u} = (J')^{-1}$, and $c = \log(J') + S + y - \log(-V(0))$. According to Ito's lemma, the drift of compensation is

$$
\begin{aligned}
& \frac{J''}{J'}\left(r(-1-\hat{u})+\frac{1}{2}\hat{\Delta}^2-a_h\right)+r(-1-\hat{u})+\frac{1}{2}\hat{\Delta}^2+\frac{1}{2}\frac{J'''J'-J''^2}{J'^2}(\hat{\Delta}-\sigma)^2 \\
= & \frac{J''}{J'}\left(r(-1-\hat{u})+\frac{1}{2}\hat{\Delta}^2-a_h\right)+r(-1-\hat{u})+\frac{1}{2}\hat{\Delta}^2+\frac{1}{2}\frac{J'''}{J'}(\hat{\Delta}-\sigma)^2-\frac{1}{2}\frac{J''^2}{J'^2}(\hat{\Delta}-\sigma)^2 \\
= & \frac{J''}{J'}\left(r(-1-\hat{u})+\frac{1}{2}\hat{\Delta}^2-a_h\right)+r(-1+\frac{1}{J'})+\frac{1}{2}\frac{J'''}{J'}(\hat{\Delta}-\sigma)^2 \\
= & \left(J''\left(r(-1-\hat{u})+\frac{1}{2}\hat{\Delta}^2-a_h\right)+r+\frac{1}{2}J'''(\hat{\Delta}-\sigma)^2-rJ'\right)J'^{-1},
\end{aligned}
$$

where the second equality follows from $\hat{\Delta}=\frac{\sigma J''}{J'+J''}$ and $\hat{\Delta}^2=\frac{J''^2}{J'^2}(\hat{\Delta}-\sigma)^2$. Differentiating the HJB equation with respect to S and applying the Envelope theorem yield

$$
rJ'=r+J''\left(r(-1-\hat{u})+\frac{1}{2}\hat{\Delta}^2-a_h\right)+\frac{1}{2}J'''(\hat{\Delta}-\sigma)^2.
$$

Therefore, the drift of compensation is zero. The sensitivity of compensation is

$$
\frac{J'+J''}{J'}(\hat{\Delta}-\sigma)+\sigma=\frac{J'+J''}{J'}\left(-\frac{J'\sigma}{J'+J''}\right)+\sigma=0.
$$

Finally, we show that consumption is nondecreasing in any time interval I such that $S_t<S^*$ for all $t\in I$. Suppose by contradiction that $c_t>c_s$ for some $t,s\in I$, $t<s$. Let ϵ be a small positive number and consider an alternative consumption plan $(\tilde{c}_t,\tilde{c}_s)$ given by $u(\tilde{c}_t)=u(c_t)-\epsilon$ and $u(\tilde{c}_s)=u(c_s)+\epsilon$. We will show that $(\tilde{c}_t,\tilde{c}_s)$ is better than (c_t,c_s), contradicting the optimality of the contract. The variation $(\tilde{c}_t,\tilde{c}_s)$ not only reduces the principal's cost (because the agent is risk averse) but also maintains the quitting constraint (because the agent's continuation value at t is unchanged, and her continuation value at s is increased). When $\epsilon>0$ is small, this variation does not violate any IC constraints because the constraints are slack in the time interval I. ■

Verification of optimality of high effort

Lemma A.11 *Under Assumption 1, it is optimal to implement high effort for all $S\geq 0$.*

Proof To show that low effort is suboptimal, we need to verify that

$$
\begin{aligned}
& \min_{\hat{u},\hat{\Delta}\geq-\hat{u}\beta} r(-\log(-\hat{u}))+J'(S)\left(r(-1-\hat{u})+\frac{1}{2}\hat{\Delta}^2-a_h\right)+\frac{1}{2}J''(S)(\hat{\Delta}-\sigma)^2 \\
\leq & \min_{\hat{u},\hat{\Delta}\leq-\hat{u}\beta} r(-\log(-\hat{u}))+J'(S)\left(r(-1-\hat{u}\phi)+\frac{1}{2}\hat{\Delta}^2-a_l\right)+\frac{1}{2}J''(S)(\hat{\Delta}-\sigma)^2. \quad (59)
\end{aligned}
$$

The following two steps verify (59) for $S\leq S^*$ and for $S>S^*$, respectively.

First, if $S \le S^*$, then the IC constraint $\hat{\Delta} \ge -\hat{u}\beta$ is slack according to Proposition 4. Inequality (59) is equivalent to

$$J'(S)(a_h - a_l) \ge r\log(\phi^{-1}),$$

which follows from $J'(S) \ge J'(0) > \frac{\kappa}{\kappa+1}$ and Assumption 1.

Second, if $S > S^*$ (i.e., the IC constraint binds), then $\frac{\sigma J' J''}{J' + J''} \le \beta$. We have

$$\begin{aligned}
& \min_{\hat{u},\hat{\Delta}\ge -\hat{u}\beta} r(-\log(-\hat{u})) + J'\left(r(-1-\hat{u}) + \frac{1}{2}\hat{\Delta}^2 - a_h\right) + \frac{1}{2}J''(\hat{\Delta}-\sigma)^2 \\
\le\ & \left(r(-\log(-\hat{u})) + J'\left(r(-1-\hat{u}) + \frac{1}{2}\hat{\Delta}^2 - a_h\right) + \frac{1}{2}J''(\hat{\Delta}-\sigma)^2\right)\Big|_{\hat{u}=\frac{-1}{J'},\hat{\Delta}=\frac{\beta}{J'}} \\
=\ & \big(r(-\log(-\hat{u})) + J'\left(r(-1-\hat{u}) - a_h\right)\big)\big|_{\hat{u}=\frac{-1}{J'}} + \frac{1}{2J'^2}(J'\beta^2 + J''(\sigma J' - \beta)^2) \\
\le\ & \big(r(-\log(-\hat{u})) + J'\left(r(-1-\hat{u}) - a_h\right)\big)\big|_{\hat{u}=\frac{-1}{J'}} + \frac{1}{2}\beta\sigma,
\end{aligned}$$

where the last inequality follows from $\sigma J' \ge \frac{r\sigma\log(\phi^{-1})}{a_h - a_l} \ge \frac{r\sigma(1-\phi)}{a_h-a_l} = \beta$ and

$$\begin{aligned}
\beta\sigma J'^2 - J'\beta^2 - J''(\sigma J' - \beta)^2 &= (\sigma J' - \beta)(J'\beta - J''(\sigma J' - \beta)) \\
&= (\sigma J' - \beta)(J' + J'')\left(\beta - \frac{\sigma J' J''}{J' + J''}\right) \ge 0.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
& \big(r(-\log(-\hat{u})) + J'\left(r(-1-\hat{u}) - a_h\right)\big)\big|_{\hat{u}=\frac{-1}{J'}} + \frac{1}{2}\beta\sigma \\
\le\ & \min_{\hat{u}} r(-\log(-\hat{u})) + J'\left(r(-1-\hat{u}\phi) - a_h\right) + r\log(\phi^{-1}) + \frac{1}{2}\beta\sigma \\
\le\ & \min_{\hat{u},\hat{\Delta}\le -\hat{u}\beta} r(-\log(-\hat{u})) + J'\left(r(-1-\hat{u}\phi) + \frac{1}{2}\hat{\Delta}^2 - a_l\right) + \frac{1}{2}J''(\hat{\Delta}-\sigma)^2 \\
& + J'(-a_h + a_l) + r\log(\phi^{-1}) + \frac{1}{2}\beta\sigma \\
\le\ & \min_{\hat{u},\hat{\Delta}\le -\hat{u}\beta} r(-\log(-\hat{u})) + J'\left(r(-1-\hat{u}\phi) + \frac{1}{2}\hat{\Delta}^2 - a_l\right) + \frac{1}{2}J''(\hat{\Delta}-\sigma)^2,
\end{aligned}$$

where the last inequality follows from Assumption 1. Thus (59) is verified. ■

Proof of Proposition 5

We start with the following auxiliary lemma:

Lemma A.12 *$J'''(S) < 0$ for all $S \ge 0$. Further,* $\lim_{S\to\infty} J''(S) = 0$.

Proof The proof of Proposition 4 has shown that $\frac{J'(S)J''(S)}{J'(S)+J''(S)} = \frac{1}{\frac{1}{J'(S)}+\frac{1}{J''(S)}}$ is strictly decreasing in S when $S < S^*$. Hence either $J'(S)$ or $J''(S)$ must be strictly decreasing. Since $J'(S)$ increases with S, $J''(S)$ strictly decreases with S when $S < S^*$. If $J''(S)$ is not globally decreasing, then there is a $\bar{S} \geq S^*$ at which $J'''(\bar{S}) = 0$. When the IC constraint binds at $S > S^*$, we have $\hat{\Delta} = -\hat{u}\beta$ and the HJB equation takes the form of

$$\begin{aligned} rJ(S) &= rS - r\log(-V(0)) \\ &\quad + \min_{\hat{u}} \left\{ -r\log(-\hat{u}) + J'(S)\left(r(-1-\hat{u}) + \frac{1}{2}(-\hat{u}\beta)^2 - a_h\right) + \frac{1}{2}J''(S)(-\hat{u}\beta - \sigma)^2 \right\}. \end{aligned}$$

The first-order condition for the optimal $\hat{u}$ is

$$rc'(\hat{u}) + J''(S)\sigma\beta = rJ'(S) + (J'(S) + J''(S))\beta^2(-\hat{u}). \tag{60}$$

Because J' increases with S while J'' is stationary at $S = \bar{S}$, equation (60) implies that $(-\hat{u})$ and $\hat{\Delta}$ decrease with S, when S is close to $\bar{S}$. Differentiating the HJB equation yields

$$0 = r + J''\left(r(-1-\hat{u}) + \frac{1}{2}\hat{\Delta}^2 - a_h\right) + \frac{1}{2}J'''(\sigma - \hat{\Delta})^2 - rJ'.$$

Because the term $J''\left(r(-1-\hat{u}) + \frac{1}{2}\hat{\Delta}^2 - a_h\right) - rJ'$ decreases with $S \in (\bar{S} - \epsilon, \bar{S} + \epsilon)$ for a small ϵ, $J'''(S) < 0$ for $S \in (\bar{S} - \epsilon, \bar{S})$ and $J'''(S) > 0$ for $S \in (\bar{S}, \bar{S} + \epsilon)$. Because of these two inequalities, J''' cannot be zero again for any $S > \bar{S}$. That is, $J''' > 0$ for all $S > \bar{S}$. Then J'' increases with S and J' will reach one eventually, a contradiction. ■

Now we can prove the proposition.

First, we show that $(-\hat{u})$ and $\hat{\Delta}$ decrease with S. If $S \leq S^*$, then $-\hat{u} = \frac{1}{J'(S)}$ and $\hat{\Delta} = \frac{\sigma J''}{J'+J''}$ decrease with S, because J' increases and J'' decreases with S. If $S \geq S^*$, then $(-\hat{u})$ and $\hat{\Delta}$ decrease with S, because in the first-order condition (60), J' increases and J'' decreases with S, and $\sigma > \beta(-\hat{u})$. Further, because $\lim_{S\to\infty} J'(S) = 1$ and $\lim_{S\to\infty} J''(S) = 0$, the first-order condition (60) approaches condition (33), which means that $\lim_{S\to\infty}(-\hat{u}) = \rho$ and $\lim_{S\to\infty} \hat{\Delta} = \rho\beta$.

Second, we show the properties of the drift and the sensitivity of S. That $\alpha(S) = r(-1 - \hat{u}) + \frac{1}{2}\hat{\Delta}^2 - a_h$ and $\xi(S) = \hat{\Delta} - \sigma$ are decreasing in S is because $-\hat{u}$ and $\hat{\Delta}$ decrease with S. That $\alpha(0) > 0$ follows from $\hat{\Delta}(0) = \sigma$, $-\hat{u}(0) = \frac{1}{J'(0)}$, and $J'(0) < \frac{r}{r+a_h-\frac{1}{2}\sigma^2}$ in Lemma A.9. That $\lim_{S_t\to\infty} \alpha(S_t) = -\mu - a_h$ follows from $\lim_{S\to\infty}(-\hat{u}) = \rho$, $\lim_{S\to\infty} \hat{\Delta} = \rho\beta$, and the definition of μ. That $\lim_{S\to\infty} \zeta(S) = \rho\beta - \sigma$ follows from $\lim_{S\to\infty} \hat{\Delta} = \rho\beta$. ■

Proof of Theorem 2

The proof proceeds in two steps: the first step constructs a diffusion process x on $(-\infty, \infty)$, the second step shows that x has an invariant distribution, which implies that S has an invariant distribution too.

First, the equilibrium dynamics of the state variable S_t in the full-commitment model can be obtained by substituting the optimal policies $-\hat{u}_t = \rho$ and $\hat{\Delta}_t = \rho\beta$ into (21):

$$dS_t = -(\mu + a_h)\,dt - (\sigma - \rho\beta)\,dz_t^a. \tag{61}$$

(i) Construct a function $f : [J'(0), \infty) \to [0, \infty)$ such that $f'(x) > 0$ for $x > J'(0)$, $f''(x)$ is continuous for $x \geq J'(0)$, and $f(x) = S(x)$ for $x \approx J'(0)$ and $f(x) = x$ for large x. Here $S(x)$ denotes the mapping from $x = J'(S)$ to S, from the change of variable in Lemma A.5. We can extend the domain of f to $(-\infty, \infty)$ by defining $f(x) \equiv f(2J'(0) - x)$ for $x < J'(0)$. Because $S'(x)|_{x=J'(0)} = 0$, the left derivative and right derivative of f are equal at $x = 0$. Hence, f is still continuously differentiable after the extension.

(ii) Construct a diffusion process for $x \in (-\infty, \infty)$ as follows. Because S is a diffusion process, so is $x = f^{-1}(S)$ whenever $x > J'(0)$. The drift $\bar{\alpha}(x)$ and sensitivity $\bar{\zeta}(x)$ of x are, respectively,

$$\begin{aligned}\bar{\alpha}(x) &= (f^{-1})'(S)\alpha(S) + \frac{1}{2}(f^{-1})''(S)(\zeta(S))^2,\\ \bar{\zeta}(x) &= (f^{-1})'(S)\zeta(S),\end{aligned}$$

where $S = f(x)$. Symmetrically, if $x < J'(0)$, then $x = 2J'(0) - f^{-1}(S)$ is also a diffusion process.

Second, for S to have an invariant distribution it is sufficient to show that x has an invariant distribution. To show that x has an invariant distribution on $(-\infty, \infty)$ we verify the sufficient conditions in Karatzas and Shreve (1991, Exercise 5.40, page 352).

(i) Nondegeneracy. The sensitivity $\bar{\zeta}(x) \neq 0$ at $x > J'(0)$ because $f'(x) > 0$ for $x > J'(0)$ and $\zeta(S) \neq 0$ for $S > 0$. Although $\zeta(0) = 0$, $\bar{\zeta}(J'(0)) \neq 0$ because $f^{-1}(S) = J'(S)$ for $S \approx 0$ and

$$\lim_{x \downarrow J'(0)} \bar{\zeta}(x) = \lim_{S \downarrow 0} J''(S)(\hat{\Delta} - \sigma) = \lim_{S \downarrow 0} \frac{J''(S)J'(S)}{J'(S) + J''(S)} > 0.$$

The sensitivity $\bar{\zeta}(x) \neq 0$ at $x < J'(0)$ due to symmetry.

(ii) Local integrability. Because $\zeta(x)$ is continuous in x and is always nonzero, it is bounded away from zero. That is, there exists $\epsilon > 0$ such that $(\zeta(x))^2 \geq \epsilon$ for all x.

(iii) $p(-\infty) = -\infty$ and $p(\infty) = \infty$, where the *scale function* $p(x)$ is defined as

$$p(x) \equiv \int_c^x \exp\left(-2\int_c^\xi \frac{\bar{\alpha}(\theta)}{\bar{\zeta}(\theta)^2} d\theta\right) d\xi,$$

where c is a fixed number. We will only show $p(\infty)=\infty$ as the proof for $p(-\infty)=-\infty$ is similar. Since $f(\theta)=\theta$ for large θ, $\lim_{\theta\to\infty}\frac{\bar{\alpha}(\theta)}{\bar{\zeta}(\theta)^2}=\lim_{\theta\to\infty}\frac{\alpha(\theta)}{\zeta(\theta)^2}=\frac{-\mu-a_h}{\sigma^2}<0$, where the inequality follows from the assumption $-\mu-a_h<0$.. Therefore, $\lim_{\xi\to\infty}-2\int_c^\xi\frac{\bar{\alpha}(\theta)}{\bar{\zeta}(\theta)^2}d\theta=\infty$, and $\lim_{x\to\infty}p(x)=\infty$.

(iv) $m(-\infty,\infty)<\infty$, where the *speed measure* m is defined as

$$m(dx)\equiv\frac{2dx}{p'(x)\bar{\zeta}(x)^2}.$$

Because $\lim_{\theta\to\infty}\frac{\bar{\alpha}(\theta)}{\bar{\zeta}(\theta)^2}=\frac{-\mu-a_h}{\sigma^2}<0$ and $\lim_{\theta\to\infty}\bar{\zeta}(\theta)^2=\sigma^2$, there is a large $\bar{x}$, such that $\frac{\bar{\alpha}(\theta)}{\bar{\zeta}(\theta)^2}<\frac{-\mu-a_h}{2\sigma^2}$ and $\bar{\zeta}(\theta)^2>\frac{\sigma^2}{2}$ for $\theta\geq\bar{x}$. Hence, if $x\geq\bar{x}$, then

$$\begin{aligned}p'(x)\bar{\zeta}(x)^2 &= \exp\left(-2\int_c^x\frac{\bar{\alpha}(\theta)}{\bar{\zeta}(\theta)^2}d\theta\right)\bar{\zeta}(x)^2\\ &\geq \exp\left(-2\int_c^{\bar{x}}\frac{\bar{\alpha}(\theta)}{\bar{\zeta}(\theta)^2}d\theta\right)\exp\left(\frac{\mu+a_h}{\sigma^2}(x-\bar{x})\right)\frac{\sigma^2}{2},\end{aligned}$$

which implies that $m([\bar{x},\infty))=\int_{\bar{x}}^\infty\frac{2dx}{p'(x)\bar{\zeta}(x)}dx$ is finite. That $m((-\infty,2J'(0)-\bar{x}])<\infty$ follows from symmetry.

■

Proof of Proposition 6

We start with the following auxiliary lemma:

Lemma A.13 *Let* $S_1=1-\log(1+\frac{1-e^{-\kappa}}{\kappa})$ *and* $S_2=2-\log(1+\frac{1-e^{-2\kappa}}{\kappa})$. *For large* a_h, $r(-1+\frac{1}{J'_{FI}(S_1)})+\frac{1}{2}\sigma^2-a_h<-\frac{a_h}{3}$ *and* $S_1<S_2<S^*$.

Proof First, we compute $J'_{FI}(S_1)$ in the full-information model. From the proof of Proposition 3, we know that $u_t=\frac{\kappa+1}{\kappa}V_{FI}(m_t)$ and $W_t=(1+\frac{1-e^{-\kappa(m_t-y_t)}}{\kappa})V_{FI}(m_t)$. Therefore, $\hat{u}(S_t)=\frac{u_t}{-W_t}=-\frac{\frac{\kappa+1}{\kappa}}{1+\frac{1-e^{-\kappa(m-y)}}{\kappa}}$. The first-order condition in the HJB equation implies $\hat{u}(S)=-\frac{-1}{J'_{FI}(S)}$, which together with $m-y=1$ at S_1 imply that

$$r\left(-1+\frac{1}{J'_{FI}(S_1)}\right)=r\left(-1+\frac{\frac{\kappa+1}{\kappa}}{1+\frac{1-e^{\kappa(y-m)}}{\kappa}}\right)=\frac{re^{-\kappa}}{\kappa+1-e^{-\kappa}}.$$

It follows from $\lim_{a_h\to\infty}a_h\kappa=r$ that

$$\lim_{a_h\to\infty}\frac{\frac{re^{-\kappa}}{\kappa+1-e^{-\kappa}}}{a_h}=\lim_{a_h\to\infty}\frac{re^{-\kappa}}{(\kappa+1-e^{-\kappa})a_h}=\frac{1}{2}.$$

Therefore, $r\left(-1+\frac{1}{J'_{FI}(S_1)}\right)+\frac{1}{2}\sigma^2-a_h<-\frac{a_h}{3}$ for large a_h.

Second, $S_1 < S_2$ because $\lim_{a_h\to\infty} S_1 - S_2 = -1 - \lim_{\kappa\to 0}\log(1+\frac{1-e^{-\kappa}}{\kappa}) + \lim_{\kappa\to 0}\log(1+\frac{1-e^{-2\kappa}}{\kappa}) = -1+\log(\frac{3}{2}) < 0$.

Third, $S_2 < S^*_{FI}$, where S^*_{FI} denotes the smallest S at which the IC constraint is violated in the full-information model. At S_2, $m-y=2$. It follows from $e^{\kappa(-2)} > \frac{r(1-\phi)}{a_h-a_l}\frac{\kappa+1}{\kappa}$ that $S_2 < S^*_{FI}$ for large a_h.

Fourth, $S^*_{FI} < S^*$. We show that $J'(S^*_{FI}) > J'_{FI}(S^*_{FI})$ and $J''(S^*_{FI}) > J''_{FI}(S^*_{FI})$. An argument similar to part (i) in the proof of Lemma A.9 shows the former. To see the latter, suppose by contradiction $J''(S^*_{FI}) \le J''_{FI}(S^*_{FI})$. The HJB equation for J_{FI} is

$$\begin{aligned} rJ_{FI}(S^*_{FI}) + r\log(-V_{FI}(0)) &= r(S^*_{FI}-\log(-\hat{u})) + J'_{FI}(S^*_{FI})\left(r(-1-\hat{u})+\frac{1}{2}\hat{\Delta}^2-a_h\right) \\ &\quad +\frac{1}{2}J''_{FI}(S^*_{FI})(\hat{\Delta}-\sigma)^2. \end{aligned}$$

Hence $J'(S^*_{FI}) > J'_{FI}(S^*_{FI})$, $J''(S^*_{FI}) \le J''_{FI}(S^*_{FI})$, and $r(-1-\hat{u})+\frac{1}{2}\hat{\Delta}^2-a_h<0$ imply that

$$\begin{aligned} & rJ_{FI}(S^*_{FI}) + r\log(-V_{FI}(0)) \\ = \;& r(S^*_{FI}-\log(-\hat{u})) + J'_{FI}(S^*_{FI})\left(r(-1-\hat{u})+\frac{1}{2}\hat{\Delta}^2-a_h\right)+\frac{1}{2}J''_{FI}(S^*_{FI})(\hat{\Delta}-\sigma)^2 \\ > \;& r(S^*_{FI}-\log(-\hat{u})) + J'(S^*_{FI})\left(r(-1-\hat{u})+\frac{1}{2}\hat{\Delta}^2-a_h\right)+\frac{1}{2}J''(S^*_{FI})(\hat{\Delta}-\sigma)^2 \\ \ge \;& rJ(S^*_{FI}) + r\log(-V(0)), \end{aligned}$$

which contradicts $J_{FI}(S)+\log(-V_{FI}(0)) \ge J(S)+\log(-V(0))$ for all $S\ge 0$. That $S^*_{FI} < S^*$ follows from

$$\frac{\sigma J'(S^*_{FI})J''(S^*_{FI})}{J'(S^*_{FI})+J''(S^*_{FI})} > \frac{\sigma J'_{FI}(S^*_{FI})J''_{FI}(S^*_{FI})}{J'_{FI}(S^*_{FI})+J''_{FI}(S^*_{FI})} = \beta.$$

■

Now we can prove the proposition.

Because the trend of S is negative in $[S_1,\infty)$, the derivative of the scale function, $p'(S)$, is strictly increasing in S. Further,

$$\begin{aligned} \left(\log(p'(S))\right)' &= -2\left(\frac{r(-1+\frac{1}{J'(S)})+\frac{1}{2}\hat{\Delta}(S)^2-a_h}{(\hat{\Delta}(S)-\sigma)^2}\right) \\ &\ge -2\left(\frac{r(-1+\frac{1}{J'_{FI}(S_1)})+\frac{1}{2}\sigma^2-a_h}{(\hat{\Delta}(S)-\sigma)^2}\right) \ge \frac{2a_h}{3\sigma^2}, \quad \text{for } S\ge S_1, \end{aligned}$$

where the first inequality follows from $J'(S) \geq J'(S_1) > J'_{FI}(S_1)$ and the second inequality follows from $r(-1 + \frac{1}{J'_{FI}(S_1)}) + \frac{1}{2}\sigma^2 - a_h < -\frac{a_h}{3}$, which is shown in Lemma A.13. This implies that $p'(S) \geq p'(S^*) \exp(\frac{2a_h}{3\sigma^2}(S - S^*))$ for $S \geq S^*$. We have

$$\frac{m[S^*,\infty)}{m[S_1,S_2]} = \frac{\int_{S^*}^{\infty} \frac{1}{p'(S)(\hat{\Delta}(S)-\sigma)^2} dS}{\int_{S_1}^{S_2} \frac{1}{p'(S)(\hat{\Delta}(S)-\sigma)^2} dS} \leq \frac{\int_{S^*}^{\infty} \frac{1}{p'(S)} dS}{\int_{S_1}^{S_2} \frac{1}{p'(S)} dS},$$

which follows from $\hat{\Delta}(S) > \hat{\Delta}(\tilde{S})$ for all $S < S^* < \tilde{S}$. This inequality is shown by $\hat{\Delta}(S) = \frac{J''(S)}{J'(S)+J''(S)} = \frac{J'(S)J''(S)}{J'(S)+J''(S)}(J'(S))^{-1} > \beta(J'(S))^{-1} \geq \beta(J'(\tilde{S}))^{-1} = \beta(-\hat{u}(\tilde{S})) = \hat{\Delta}(\tilde{S})$. Further,

$$\frac{\int_{S^*}^{\infty} \frac{1}{p'(S)} dS}{\int_{S_1}^{S_2} \frac{1}{p'(S)} dS} \leq \frac{\int_{S^*}^{\infty} \frac{1}{p'(S)} dS}{(S_2 - S_1)\frac{1}{p'(S^*)}} \leq \frac{\int_{S^*}^{\infty} \frac{1}{\exp(\frac{2a_h}{3\sigma^2}(S-S^*))} dS}{S_2 - S_1} = \frac{1}{\frac{2a_h}{3\sigma^2}(S_2 - S_1)}.$$

Hence $\lim_{a_h \to \infty} \pi([S^*, \infty)) = \lim_{a_h \to \infty} \frac{m[S^*,\infty)}{m[0,\infty)} \leq \lim_{a_h \to \infty} \frac{m[S^*,\infty)}{m[S_1,S_2]} = 0.$ ■

Properties of the cost function J_{FI} and dynamics of the state variable S_t in the model with full information

Lemma B.1 *In the model with full information,*

(i) J'_{FI} *is everywhere positive and strictly increasing with*

$$J'_{FI}(0) = \frac{\kappa}{\kappa+1} \quad \text{and} \quad \lim_{S_t \to \infty} J'_{FI}(S_t) = 1.$$

(ii) J''_{FI} *is everywhere positive and strictly decreasing with*

$$J''_{FI}(0) = \infty \quad \text{and} \quad \lim_{S_t \to \infty} J''_{FI}(S_t) = 0.$$

(iii) The drift of the state variable, α*, is strictly decreasing with*

$$\alpha(0) = \frac{1}{2}(\kappa+1)\sigma^2 > 0 \quad \text{and} \quad \lim_{S_t \to \infty} \alpha(S_t) = -a_h.$$

(iv) The sensitivity of the state variable, ζ*, is everywhere negative and strictly decreasing with*

$$\zeta(0) = 0 \quad \text{and} \quad \lim_{S_t \to \infty} \zeta(S_t) = -\sigma.$$

Proof It is useful to derive policies $\hat{u}(S_t)$ and $\hat{\Delta}(S_t)$ as functions of m_t and y_t. From the proof of Proposition 3, we know that $u_t = \frac{\kappa+1}{\kappa} V_{FI}(m_t)$, $\Delta_t = -V_{FI}(m_t)e^{-\kappa(m_t - y_t)}\sigma$ and $W_t =$

$(1+\frac{1-e^{-\kappa(m_t-y_t)}}{\kappa})V_{FI}(m_t)$. Therefore,

$$\begin{aligned}\hat{u}(S_t) &= \frac{u_t}{-W_t} = -\frac{\frac{\kappa+1}{\kappa}}{1+\frac{1-e^{-\kappa(m-y)}}{\kappa}},\\ \hat{\Delta}(S_t) &= \frac{\Delta_t}{-W_t} = \frac{e^{-\kappa(m_t-y_t)}}{1+\frac{1-e^{-\kappa(m_t-y_t)}}{\kappa}}\sigma.\end{aligned}$$

This implies that $\hat{u}(S_t)$ increases and $\hat{\Delta}(S_t)$ decreases in S_t. Further, $\hat{u}(0) = -\frac{\kappa+1}{\kappa}$, $\hat{\Delta}(0) = \sigma$, $\lim_{S_t\to\infty}\hat{u}(S_t) = \lim_{(m_t-y_t)\to\infty}\hat{u}(S_t) = -1$, and $\lim_{S_t\to\infty}\hat{\Delta}(S_t) = \lim_{(m_t-y_t)\to\infty}\hat{\Delta}(S_t) = 0$.

(i) Since $J'_{FI}(S_t) = (-\hat{u}(S_t))^{-1}$, the property of $J'_{FI}(S_t)$ follows from that of $\hat{u}(S_t)$ in the above.

(ii) It follows from $J'_{FI}(S_t) = (-\hat{u}(S_t))^{-1} = (1+\frac{1-e^{-\kappa(m_t-y_t)}}{\kappa})/\frac{\kappa+1}{\kappa}$ and $S_t = m_t - y_t - \log(\kappa + 1 - e^{-\kappa(m_t-y_t)})$ that

$$J''_{FI}(S_t) = \frac{\kappa}{\kappa+1}\frac{e^{-\kappa(m_t-y_t)}}{1-\frac{e^{-\kappa(m_t-y_t)}\kappa}{\kappa+1-e^{-\kappa(m_t-y_t)}}} = \frac{\kappa}{\kappa+1}\frac{1}{e^{\kappa(m_t-y_t)}-\frac{\kappa}{\kappa+1-e^{-\kappa(m_t-y_t)}}},$$

which decreases in $m_t - y_t$. If $S_t = 0$, then $m_t - y_t = 0$ and clearly $J''_{FI}(0) = \infty$. Moreover, $\lim_{S_t\to\infty} J''_{FI}(S_t) = \lim_{m_t-y_t\to\infty} J''_{FI}(S_t) = 0$.

(iii) It follows from $\alpha(S_t) = r(-1-\hat{u}(S_t)) + \frac{1}{2}(\hat{\Delta}(S_t))^2 - a_h$ that $\alpha(S_t)$ decreases in S_t. Further,

$$\begin{aligned}\alpha(0) &= r(-1+\frac{\kappa+1}{\kappa}) + \frac{1}{2}\sigma^2 - a_h = \frac{1}{2}(\kappa+1)\sigma^2,\\ \lim_{S_t\to\infty}\alpha(S_t) &= r(-1+1) + \frac{1}{2}0^2 - a_h = -a_h.\end{aligned}$$

(iv) It follows from $\zeta(S_t) = \hat{\Delta}(S_t) - \sigma$ that $\zeta(S_t)$ decreases in S_t. Further,

$$\zeta(0) = \hat{\Delta}(0) - \sigma = 0, \quad \lim_{S_t\to\infty}\zeta(S_t) = 0 - \sigma = -\sigma.$$

■

Discussion. The quitting constraint is the only friction in the full-information version of our model. If this friction were absent, the contracting environment would be the so-called first best: firms would fully insure workers against fluctuations in their productivity by giving them permanently constant compensation and workers would be committed to never quitting or shirking. In the first best, W_t is constant, so, as evident from (19), the dynamics of the state variable S_t reduce to $dS_t = -dy_t$, which means that $\alpha(S_t) = -a_h$ and $\zeta(S_t) = -\sigma$ at all S_t. With the worker's compensation constant, the firm's profit simply follows the random changes in the output produced by the worker. The first-best cost function, denoted as J_{FB}, therefore satisfies $J'_{FB}(S_t) = 1$, as a larger drift of the worker's output process would reduce the firm's cost one-to-one.[49] Also, since firms are risk-neutral and never run into quitting or incentive constraints in the first best, they are indifferent to volatility in S_t. This means that $J''_{FB}(S_t) = 0$ for all S_t.[50]

Lemma B.1 shows that the equilibrium cost function and the dynamics of the state variable in the model with the quitting constraint converge to the first best when slackness S_t in the quitting constraint becomes large. This convergence is intuitive. When S_t is large, the expected time until the quitting constraint binds again is large, and so the equilibrium contract (26) is expected to provide full insurance to the worker far into the future. Since the equilibrium contract at this point approximates the first-best contract very closely, its cost is close to the first-best cost function.

On the other extreme, when the quitting constraint binds (i.e., at $S_t = 0$), ensuring that it continues to be satisfied under all realizations of the shock to the worker's productivity is only possible if, first, there is no volatility in S_t at $S_t = 0$, and, second, the drift of S_t at $S_t = 0$ is nonnegative. The optimal contract, as we see in Lemma B.1, does induce the sensitivity $\zeta(0) = 0$, which extinguishes all volatility in S_t at the boundary. Consistently, $J''_{FI}(0) = \infty$, which reflects the fact that the firm is infinitely averse to the volatility in S_t when the quitting constraint binds, as any nonzero volatility would lead to a violation of the quitting constraint with probability one immediately after S_t hits zero.

Note that the absence of volatility of S_t means that the volatility of the worker's continuation value inside the contract is the same as the volatility of her outside option, which means that locally at $S_t = 0$ the firm cannot provide any insurance to the worker. To avoid violating the quitting constant, clearly, the drift of S_t at $S_t = 0$ must be nonnegative. A strictly positive drift of S_t at $S_t = 0$ is beneficial in that it relaxes the quitting constraint, which allows the firm to provide insurance to the worker as soon as S_t becomes strictly positive. But positive drift in S_t is also costly because in order to obtain it the contract must back-load compensation and produce a strictly positive drift in the worker's continuation value W_t. Positive drift in W_t is costly as it means that intertemporal smoothing of the worker's consumption is poor. (Recall that drift of W_t at the first best is zero.) The optimal drift $\alpha(0)$ given in the above lemma is the outcome of balancing this trade-off. It is strictly positive, so zero is a reflecting rather than absorbing barrier for the state variable and insurance is provided to the worker. Its size is limited, however, by the intertemporal inefficiency of excessive compensation back-loading. Consistently, $J'_{FI}(0) = \frac{\kappa}{\kappa+1} < 1 = J'_{FB}(0)$ reflects the fact that positive drift of S_t has a benefit in the full-information-limited-enforcement model that it does not have in the first best: it helps relax the quitting constraint. As a consequence, the firm is less averse to drift in S_t than it is

[49]Recall from (23) that the first derivative of the cost function represents the impact of the state variable's drift on the firm's total cost. In the first best, the drift of the state variable is the negative of the drift of the worker's output.

[50]Recall again from (23) that the second derivative of the cost function represents the impact of the state variable's volatility on the firm's total cost.

at the first best, which means that J'_{FI} is everywhere smaller than $J'_{FB} \equiv 1$.

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