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# Asset Trading and Valuation with Uncertain Exposure

WP 14-05

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**Abstract**

This paper considers an asset market where investors have private information not only about asset payoffs, but also about their own exposure to an aggregate risk factor. In equilibrium, rational investors disagree about asset payoffs: Those with higher exposure to the risk factor are (endogenously) more optimistic about claims on the risk factor. Thus, information asymmetry limits risk sharing and trading volumes. Moreover, uncertainty about exposure amplifies the effect of aggregate exposure on asset prices, and can thereby help explain the excess volatility of prices and the predictability of excess returns.

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# 1 Introduction

Recent events have renewed interest in the role of heterogeneous beliefs for asset prices. Two prominent literatures model speculative trading. One studies the interaction of investors with (exogenously given) heterogenous beliefs about asset payoffs in the presence of trading frictions, in the tradition of Harrison and Kreps (1978). Recent work in this vein emphasizes shifts in wealth across optimists and pessimists.

A second literature considers rational expectations equilibria with private information in the tradition of Grossman (1976). Work in this area emphasizes endogenous formation of beliefs about asset payoffs through learning from prices. At the same time, it typically makes assumptions on preferences, technology and information so that the distribution of investors' initial positions or shifts in wealth across investors do not matter for prices.

This paper considers asset markets where investors not only receive private signals about the payoffs on tradable assets, but also learn about shifts in the distribution of others' positions by observing their own position. The latter inference is relevant, for example, when there are aggregate shocks to the value of nontraded or very illiquid assets such as bank loans, private equity, or human capital. The distribution, and often even the aggregate quantity, of illiquid assets owned by participants in an asset market is arguably difficult to observe. Nevertheless, it can be important for trading and valuation of more liquid tradable assets if it affects investors' overall risk exposure, which is generally a function of their total portfolio.

Consider, as an illustration, institutional investors who trade in the (liquid) public equity market. If their (illiquid) loan or private equity positions lose value, they may conclude that there has been a drop in aggregate net worth since other institutions presumably suffered a loss too, thus increasing the overall exposure of the financial system to public equity risk. Of course, investors can look at stock price movements to try to learn the shock to aggregate exposure to equity. However, if prices are also affected by dispersed information about future equity payoffs, learning may be incomplete. The purpose of the present paper is to explore the role of aggregate exposure shocks for asset prices and trading volume in the presence of private information.

We show how, with private information about both payoffs and aggregate exposure, rational investors choose to disagree about asset payoffs in equilibrium. In particular, those investors with higher initial exposure to a risk factor will be more optimistic about claims on that risk factor, so that there is less risk sharing than under symmetric information. Moreover, uncertainty about exposure amplifies the effect of aggregate exposure shocks on prices and can thereby help explain their excess volatility; in particular, shocks to aggregate exposure can jointly generate "excessively" low prices and low trading volume.

In the absence of private information, our model works like the standard representative agent pricing model in Lucas (1978). We consider an exchange economy without trading restrictions such as short-sale or leverage constraints. There are enough assets available so investors' endowments are tradable. Preferences exhibit risk aversion and can allow for wealth effects—our leading example assumes log utility. With symmetric information, there is then a stochastic discount factor $m$, proportional to representative agent marginal utility, so the price of a tradable risk factor $\tau$ can be written in the standard way as

$$p = E\left[m\tau\right] = E\left[\tau\right]/R^f - e\ var\left(\tau\right),$$

where $R^f$ is the risk-free interest rate, the mean and variance of $\tau$ reflect investors common beliefs, and $e \equiv cov(m, \tau)/var\left(\tau\right)$. We refer to $e$ here as the *aggregate exposure* of the economy to the risk factor $\tau$. It generally depends on the distribution of initial positions and preferences, and the direct effect of shocks to aggregate exposure is to lower prices and increase risk premia, holding beliefs fixed.[1] Trading volume then follows from investors sharing risk, in the sense of equating (initially heterogenous) individual exposures to $\tau$-risk.[2]

In the presence of with private information, there are two elements in the asset-pricing formula above where heterogeneous beliefs are pertinent: information about the payoff distribution for $\tau$ (its mean and variance in the formula) and information about aggregate exposure $e$. If the price is the only public signal in the economy, then these two elements will not be revealed. For example, if the asset price is low, investors cannot know whether this is a result of dispersed private signals about payoffs on average conveying bad news, or a result of dispersed knowledge of positions on average conveying high aggregate exposure. For this reason, notice that while the direct negative effect of an aggregate exposure shock on prices remains present to its full extent, there is now also a negative indirect, or "spillover" effect on beliefs, as the exposure shock is partly mistaken for bad news.

While an aggregate exposure shock is amplified by the presence of private information, an aggregate news shock about payoffs is dampened. This is true even though it is partly

[1] We describe prices using payoffs and aggregate exposure $e$, rather than simply payoffs and the stochastic discount factor $m$ because $e$ is the particular moment that matters for information aggregation in our models. For example, fully informative equilibria exist if $e$ is constant, regardless of other properties of $m$.

[2] Since exposure depends on preferences, this need not mean that all investors hold the same portfolio.

misunderstood as an exposure shock—bad news leading to a price fall can be interpreted also as a rise in aggregate exposure. The key difference is that news about payoffs have no spillover effects on prices via exposure. Indeed, payoffs news change beliefs about aggregate exposure, but what matters for prices in our model is actual aggregate exposure, not beliefs about exposure. A key insight in our paper, therefore, is that payoff-relevant news shocks and exposure shocks are qualitatively very different in a world with private information. In particular, the price reaction to exposure shocks relative to that to news shocks about payoffs is stronger under private information.

With private information, trading volume is no longer driven only by the desire to share risk, but also by differences in beliefs, due in turn to different inferences from signals and prices. Consider, for example, an investor who starts from high initial exposure to the tradable risk factor, and thus believes that others investors have high exposure as well. This investor will endogenously become more optimistic about the asset payoff than an investor with low initial exposure. The reason is that high-exposure investors will tend to extrapolate from their own exposure to think that the aggregate exposure is high, putting downward pressure on the price; however, low-exposure investors observe the same price and Bayesian reasoning will then lead high-exposure agents to have more optimistic views about payoffs than will low-exposure agents.

The effect of aggregate exposure shocks on prices and volume helps provide an interpretation of the 2008 US financial crisis. Think of $\tau$ as the risk associated with the US housing market and of investors as financial institutions trading mortgage-backed bonds. Banks' exposure to the housing market depends on default probabilities and recovery rates for mortgage bonds. In particular, aggregate exposure is high if more mortgage bonds are "junk"—with payoffs highly dependent on house prices—rather than "high quality"—with essentially riskless payoffs. If aggregate exposure is known, then the mortgage bond market aggregates banks' private information about house values, banks agree on the course of the housing market and share exposures, between banks that have originated more subprime loans with junk payoffs and banks that have originated more high quality loans.

Consider now a shock that increases *some* individual banks' exposures to the housing market, where both the scale of the shock and the identities of the affected banks remain unknown. A trigger for such a shock might be loss of trust in rating agencies due to information about their mistakes in rating specific bonds.[3] Our model says that such a shock would generate a

[3]In addition, an aggregate exposure shock may be accompanied by bad news about the housing market, and our model allows this. Notice also in this context that the exposure shocks might be thought in terms

drop in prices to unusually low levels—below what an observer would rationalize as the present value of houses—together with a drop in trading volume. When aggregate exposure is uncertain, the price does not properly aggregate banks' dispersed information about housing and hence becomes a worse signal of things to come.

The fall in volume in our model—like the much-discussed "market freeze" in 2008—does not occur because the market stops working. Instead, risk sharing motives are counteracted by speculative motives for trade. In particular, those investors with high exposures end up viewing low quality mortgage bonds as relatively good investments—they believe low prices reflect increased aggregate exposure to the housing market rather than bad news about house prices. As a result, they prefer not to trade away their risky assets and instead held on to them.

More generally, we show how the excess responsiveness of prices to exposure increases the predictability of asset prices. In standard asset-pricing models, high prices are followed by low returns, and vice versa: time-varying risk, or aversion to risk, leads prices to be low when risk is high, at the same time implying high risk premia, i.e., high average subsequent returns. This feature is present here as well: Exposure shocks are shocks that arise from higher risk, or risk aversion, but because they are over-reacted to, predictability is more potent, both in terms of the magnitude and the fit of a typical predictability regression. In our model, moreover, dividend shocks are qualitatively different: They are under-reacted to.

Finally, our model may also help understand some more regular, and long-since noticed, features of the data such as the high risk premia for holding stock. Our economy can deliver different premia than its symmetric-information, Lucas counterpart, but here the direction of the effect depends on the wealth distribution and on details of the shock distribution. If, namely, wealth effects are important and wealthy traders are more pessimistic than poor traders, then risk premia will be large, but whether this wealth-pessimism pattern holds in the data is an open question. What the present paper at least suggests is that this mechanism is worthy of further study.

The rest of the paper is structured as follows. Section 2 briefly discusses the relevant literature and how we view our paper as adding to it. Section 3 introduces the model, which is simple in that it involves consumption at one date only but which otherwise uses rather general assumptions, and defines a competitive equilibrium under asymmetric information. Section 4 develops our notion of exposure. Above, exposure was defined using the simple asset-

of securities going from being "information-insensitive" to "information-sensitive," in the sense of Dang et al. (2013).

pricing equation as the covariance between the risk factor $\tau$ and the stochastic discount factor, normalized by the variance of the risk factor. We show that this definition represents the premium the representative agent is willing to pay to avoid a marginal increase in $\tau$-risk. We look at exposure on the individual level—before and after trading—and at aggregate exposure, and we discuss a rather general preference structure, along with conditions under which prices will only partially reveal all relevant aggregate information.

Section 5 then describes a specific example economy—our "benchmark economy," with logarithmic utility and a convenient shock structure for the distribution of exposures and payoff news across agents—and uses it to illustrate all the main results of our paper. Section 6 generalizes from the benchmark economy in different directions. One extension keeps the shock structure but broadens the utility specification to the linear risk tolerance class, without changes in the results. Another looks at more general utility functions, but then specializes the shock structure. A third extension, finally, looks at the exponential-utility, normal-shocks case typically studied in finance applications; for this case, somewhat more detailed characterization can be obtained than in the case where wealth effects are present. Section 7 concludes.

## 2 Related literature

Relative to the literature, we make several, though related, contributions. We provide a general definition of exposure that helps predict equilibrium beliefs and trading behavior given the distribution of initial exposures. We also show how shocks to aggregate exposure become relatively more important for price volatility in the presence of private information. We establish these results for economies where agents have (possibly heterogeneous) preferences in the linear risk tolerance (LRT) class that contains both CRRA and exponential utility. We further provide a numerically tractable setting with log utility, thus allowing for decreasing absolute risk aversion and wealth effects. Finally, the application of our model to the crisis episode offers an interpretation of events that is not based on financial frictions or the irrationality of investors: It only requires an aggregate exposure shock and its imperfect revelation through market prices.

The fact that aggregate exposure shocks can matter for price volatility is familiar from the literature on asset pricing in the tradition of Lucas. In a representative agent model, any shock to marginal utility that does not affect dividends will move around the exposure of the representative agent to the stock market. Examples include shocks to housing and human capital, or exogenous changes in risk aversion. Heterogeneous-agent models with incomplete markets typically allow for changes in the idiosyncratic volatility of labor income, which also changes the initial exposure of agents to stocks. In exponential-normal models, Campbell et al. (1993), Campbell and Kyle (1991), and Spiegel (1998) have examined the role of random supply of assets for volatility. In all of these setups, aggregate exposure shocks have been found to be quantitatively important to generate excess volatility of prices. Our results say that they matter for prices in the presence of private information, and also generate interesting comovement of prices and volume.

There is a large literature on rational expectations equilibria (REE) in economies with private information, following the seminal work of Radner (1967) and Lucas (1972). The fact that equilibrium involves inference from prices has made it difficult to provide general proofs of the existence of partially revealing equilibria (see Allen and Jordan (1998) for a survey of early work and Pietra and Siconolfi (2008) for some recent results).[4] DeMarzo and Skiadas (1998) have characterized existence and asset pricing properties in a class of "quasi-complete" economies with LRT preferences that subsumes many models considered earlier. Our model economies are not quasi-complete because of the presence of aggregate exposure shocks.

For asset pricing applications, a major workhorse has been the framework with exponential utility and normally distributed shocks developed by Grossman (1976), Hellwig (1980), and Admati (1985). In these models, the presence of nonrevealing equilibria is due to a random supply of assets sold by "noise traders." A net sale by noise traders can be viewed as a shock that increases the equilibrium exposure of the rational agents. It thus generates low prices and high risk premia together with *high* volume. Moreover, the presence of private signals (about payoff) increases volume: Equilibrium disagreement of rational investors leads to extra volume from speculative trades while the response of volume to noise trades is independent of the information structure. In contrast, a key feature of our model is that the presence of private signals (about both payoff and aggregate exposure) can lead to *lower* volume.

Diamond and Verrecchia (1981) consider an exponential/normal setup where agents receive stochastic endowments from which they learn about the aggregate endowment. Our exponential-normal example in Section 6.3 is a version of their model, but with a continuum of traders and "news shocks," that is, the pooled information of all agents does not reveal asset

[4]Several authors have studied the role of exogenously given heterogenous beliefs on asset prices (see for example Calvet et al., 2001, Jouini and Napp, 2007, Jouini and Napp, 2006, Detemple and Murthy, 1994). In a rational expectations equilibrium, the heterogeneity of beliefs is endogenous.

payoffs, but only serves as an imperfect signal (or "news") about those payoffs. Similar setups have been used by Ganguli and Yang (2009) and Manzano and Vives (2011) to investigate how multiplicity of equilibria depends on information acquisition incentives and the presence of public signals, respectively.

Several papers have worked out nonrevealing rational expectations equilibria outside of the exponential/normal framework.[5] At the most general level, the effects of exposure on prices and trading we emphasize here do not require nonlinearities—as we show, they arise also in the exponential/normal setting. At the same time, we find the case where exposure shifts reflect wealth effects—due to revaluation of illiquid assets for example—appealing for further applications.

Our paper also adds to a recent literature that broadens the dimensions of uncertainty beyond private signals and private endowments.[6] Indeed, our general definition of exposure clarifies that what matters for belief formation and trading is not just investors' signals and initial positions—the latter represented, for example, by endowments in an exponential/normal setting—but also their preferences. For example, in our LRT economies, uncertainty about the share of agents with high curvature in the utility function works in much the same way as uncertainty about the correlation of the initial position with the tradable risk factor—both represent aggregate exposure risk. Heterogeneity in objective functions is of interest, for example, when thinking about price formation in markets with institutional traders, where it can capture differences in financing constraints.

A related set of papers studies price volatility in economies where conditions on type distributions, preferences and trading constraints generate an active extensive margin. The asset price is then determined by a particular marginal investor type, namely a trader who is indifferent between buying and not buying the asset, and excess volatility can emerge if the payoff expectation of this marginal investor is different from (and changes more with shocks than) that of an outside observer. Prominent examples with exogenous beliefs are Scheinkman and Xiong, 2003, Hong and Stein, 2003), Fostel and Geanakoplos (2008) and Simsek (2013). Albagli

[5]Ausubel (1990) considers a two good example with asymmetric information about preferences. Barlevy and Veronesi (2000, 2003) consider risk neutral investors who face trading constraints; they point out in particular the possibility of multiple equilibria. Breon-Drish (2013) analytically works out equilibria with non-normal payoffs and CARA utility; he shows how the resulting nonlinearities in the price function help understand asset pricing anomalies. Peress (2004) considers the interaction of wealth effects in portfolio choice and information acquisition in a noisy rational expectations model.

[6]For example, Easley et al. (2012) study uncertainty about others' risk aversion. Uncertainty about others' information set is considered by Gao et al. (2013) in a static setup and by Banerjee and Green (2014) in a dynamic model with learning.

et al. (2011) study a trading game with noise traders and private information, in which prices are characterized by the (endogenous) information of a marginal investor investor. We follow instead the older REE literature and study competitive equilibria with risk averse agents and no trading constraints, so that standard Euler equations hold for all traders—in other words, every investor is marginal. Excess volatility nevertheless emerges because payoff expectations and equilibrium exposures for all investors adjust to support a price that is different from (and changes more with aggregate exposure shocks than) what an outside observer would expect.[7]

The crisis has also renewed interest in markets with asymmetric information, that is, settings where some agents have better information than others in equilibrium. For example, in Gorton and Ordoñez (2012), lenders can choose to acquire private information about collateral posted by borrowers, thus making debt an "information sensitive" security. Guerrieri and Shimer (2014) study dynamic trading in markets where sellers know more than buyers about asset quality; they investigate when relative prices and relative volume (and hence the speed of trading) across qualities helps buyers infer quality. Vayanos and Wang (2012) compare the role of information asymmetry and imperfect competition for returns. Our perspective is different from these papers in that our setup features differential, rather than asymmetric, information. In particular, our interpretation of the crisis says that many institutions are similarly uncertain about the nature of aggregate shocks, but none has information of superior quality. Our point here is not that trading constraints or information asymmetries are not relevant; we simply emphasize that crisis-like episodes do not necessarily require those features.

Recent work has also started to examine the quantitative implications of dynamic exponential/normal models with asymmetric information for asset pricing and trading. Biais et al. (2007) show that the interaction of informed and uninformed agents can explain the superior performance of mean-variance efficient portfolio strategies used by uninformed investors relative to the market. Building on the dynamic exponential-normal model proposed by Wang (1994), Albuquerque et al. (2007) show that modeling trades between informed and uninformed traders in international equity markets help understand the joint distribution of cross border equity flows and stock returns. Nimark (2011, 2012) proposed a new computational approach for dynamic linear rational expectations models and applies it to the term structure of interest

[7]In fact, as shown in Section 5.3, the price is different from what is expected by an outside observer who knows preferences and aggregate exposure. We thus obtain excess volatility of prices not only relative to expected payoff, but also relative to risk-adjusted expected payoff, where the risk adjustment uses the correct preferences and endowments, but assumes symmetric information. In other words, private information can explain not just failure of a risk neutral pricing model, but also of a representative agent model.

rates. The computational approach in this paper may be useful for further quantitative work in the future.

# 3 Model

We consider a two period exchange economy. In the first period, investors trade claims on an aggregate risk factor that is realized at date 2 when assets pay off. For example, financial institutions trade mortgage backed securities with payoffs that depend on the U.S. housing market. At date 1, there is an aggregate shock that investors learn privately: The shock affects investors' endowments, preferences, and information sets, all of which are private information. For example, institutions learn about housing market conditions through market research and their interaction with clients. As they observe a deterioration of market conditions, they reassess not only the likelihood of an overall housing bust, but also the distribution of losses they—and other institutions like them—should expect to take in such a bust.

Formally, there are two dates and a continuum of investors of measure 1. At date 1, investors trade assets that pay off at date 2. Fix a probability space $(X, \Xi, \Pr)$ on which all random variables are defined. The aggregate risk factor is a random variable $\tau$ with $var(\tau) > 0$. It is tradable, that is, investors can trade an asset with payoff $\tau$ at date 1. Investors can also trade a riskless asset that pays off one for sure. Investors thus choose payoffs from the set

$$C = \{c : \text{there are } a_c, b_c \in \mathbb{R} \text{ s.t. } c = a_c + b_c\tau\} \tag{1}$$

Any payoff $c$ is identified with a portfolio of $a_c$ units of the riskless asset and $b_c$ claims on the risk factor $\tau$.[8] Asset prices are represented by a linear function on $C$ that assigns a value to every payoff. We normalize the price of the riskless payoff to one. The value of a payoff $c$ can then be written as $a_c + b_c p$, where the parameter $p \in \Re$ is the price of a claim on $\tau$. The exact asset structure is not important, as long as there are enough assets to generate any payoff in $C$.

*Investor heterogeneity*

To allow for investor heterogeneity, we introduce a set $\Theta$ of investor types. For a generic investor, his type $\theta$ is a random variable valued in $\Theta$ that determines his endowment, preferences

[8] We restrict attention to environments in which i) consumption plans $C$ span the space generated by risk factor payoffs, or ii) the optimal consumption in the space spanned by risk factor payoffs belongs to $C$.

and information. The investor's endowment is a random variable $\omega(\theta) \in C$. Endowments are thus tradable, that is, they can be thought of as initial portfolios of tradable assets. For example, if $\tau$ is the housing market, then $\omega$ can be viewed as an initial portfolio of mortgage bonds; the coefficient $b_{\omega(\theta)}$ says how sensitive the portfolio is to a housing boom or bust.

Investor objectives are described by a utility function $u(c, \theta)$ that ranks payoffs realized at date 2. We assume that $u$ is continuously differentiable, strictly increasing and strictly concave in $c$. Here curvature in the utility function may capture risk aversion, or it may stand in for factors that determine the risk tolerance of institutions such as financing constraints or agency problems within the firm. It can also reflect the regulatory environment—for example, we would expect institutions that anticipate more generous bailout payments to act *as if* they are less risk averse.

Finally, an investor's type $\theta$ summarizes the investor's information from any source other than market prices. Two types of information are relevant here. First, there is *information about future payoffs*, that is, the risk factor $\tau$. To accommodate payoff information, we allow an investors' type $\theta$ to be correlated with $\tau$ under the probability Pr. Second, there is *information about others' positions and attitude toward risk*. To accommodate information about others, we allow types to be correlated across agents. Importantly, we allow investors to learn information about others that is not directly payoff relevant.

To illustrate, consider again the example of financial institutions trading mortgage bonds. Individual institutions collect information by engaging in market research as well as talking to clients. Through this process, they arrive at—possibly distinct—views on the future of the housing market. In other words, they learn about a key aggregate risk factor that will affect their portfolio in the future. At the same time, they learn about how sensitive their portfolio is to the housing market. For example, they might conclude that default rates will be higher than expected, and that therefore portfolio payoff will covary more with house prices. Moreover, since institutions have some knowledge of others' business practices, they may then infer something about others' likely sensitivity to the housing market also. This information is not directly helpful in order to forecast what happens to housing. However, it is useful to understand how the economy works and in particular what forces shape the currently observed bond prices.

We impose throughout that investors have rational expectations: Everybody knows the joint distribution of all random variables under Pr. Investors can thus disagree only if their types $\theta$ convey different information. Individual information sets at date 1 are denoted $I(\theta)$; they contain an investor's own type $\theta$ as well as asset prices. Prices thus not only enter the budget constraint, but also serve as a signal. Investors use Bayes rule to form a subjective belief given the information $I(\theta)$. Individual preferences over consumption plans $c$ are therefore represented by

$$U(c;\theta) = E\left[u(c;\theta)\,|I(\theta)\right] \tag{2}$$

*The agent's problem*

An individual investor chooses a consumption plan $c \in C$ to maximize expected utility $U(c;\theta)$ from (2) subject to a budget constraint. Given our assumptions on the structure of $C$, we can write the maximization problem in terms of coefficients $a_c$ and $b_c$ that determine a consumption plan. An investor thus solves

$$\begin{aligned} &\max_{a_c, b_c} E\left[u(a_c + b_c\tau;\theta)\,|I(\theta)\right] \\ &s.t. \quad a_c + b_c p = a_{\omega(\theta)} + b_{\omega(\theta)} p \end{aligned} \tag{3}$$

Taking first-order conditions with respect to $a_c$ and $b_c$ and eliminating the Lagrange multiplier on the budget constraint delivers

$$p = E\left[\tau \frac{u'(c;\theta)}{E\left[u'(c;\theta)\,|I(\theta)\right]}|I(\theta)\right] = E\left[\tau|I(\theta)\right] - cov\left(\tau, -\frac{u'(c;\theta)}{E\left[u'(c;\theta)\,|I(\theta)\right]}|I(\theta)\right). \tag{4}$$

This first order condition is a standard asset pricing equation mentioned in the introduction: The price of a risk factor equals the risk adjusted discounted expected payoff, where the stochastic discount factor is $u'(c)/E[u'(c)]$.[9] Together with the budget constraint, it determines the coefficients $a_c$ and $b_c$ that describe the optimal consumption plan. We denote the optimal plan of a type $\theta$ agent by $c^*(\theta;p)$.

*The correlation of types*

To model the correlation of types, we introduce an aggregate shock that affects the cross section of investors. Formally, before markets open at date 1, nature draws a distribution $\mu \in \Delta(\Theta)$, where $\Delta(\Theta)$ is the set of probability distributions over $\Theta$. Types are then iid conditional on $\mu$, so $\mu(\theta)$ is both the probability that an individual investor is of type $\theta$ and the fraction of investors of type $\theta$ in the population. When investors trade assets at date 1, they know their own type, but not the distribution $\mu$. As a result, the aggregate shock $\mu$ is learned

[9]Here, because there is only consumption in one period, the risk-free rate does not appear.

privately by investors through signals, rather than being learned via a public announcement, say.

As discussed above, we would like a type to reflect both private information about payoffs (the risk factor $\tau$) and private information about others that is not payoff relevant. A simple way to accommodate both types of information is to work with distributions that can be represented by exactly two parameters. The first parameter summarizes the payoff relevant information contained in $\mu$. We assume that there exists a one-dimensional statistic $\delta(\mu)$ that is sufficient for forecasting the risk factor $\tau$ given $\mu$. If agents know the distribution $\mu$, then $\delta(\mu)$ is all they care about for the purpose of predicting payoffs. We thus refer to $\delta$ as the "aggregate payoff news" available in the economy as of date 1.

Our second key parameter, $\varepsilon$, summarizes all additional information about others' endowments and preferences that is contained in $\mu$. We assume that $\varepsilon$ cannot be written as a deterministic function of $\tau$. We want some independent variation in $\varepsilon$ to capture the idea that investors learn information about others that is not directly payoff relevant. At the same time, we do not require that $\varepsilon$ is independent of $\delta$ or $\tau$. What matters is that $\varepsilon$ is not needed to forecast $\tau$ if $\delta$ is known. To sum up, we consider joint distributions of the aggregate shocks $\mu$ and $\tau$ as well as the individual types $\theta$ such that *the pair* $(\delta(\mu), \varepsilon(\mu))$ *is a sufficient statistic for forecasting an individual's type* $\theta$ given $\mu$. All investors know the joint distribution of these variables under Pr.

*Equilibrium*

Since investor types are iid conditional on $\mu$, aggregate demand for date 2 contingent claims, and hence equilibrium prices, depend only on the distribution of types $\mu$. A *rational expectations equilibrium (REE) price function* $P : \Delta(\Theta) \to \Re$ clears the market: for every $\mu$,

$$\sum_{\theta \in \Theta} \mu(\theta)\, c^*(\theta, P(\mu)) = \sum \mu(\theta)\, \omega(\theta) =: \Omega(\mu).$$

where $c^*(\theta, P(\mu))$ is the optimal solution to (3) given the price realization $P(\mu)$. Importantly, when agents form their subjective belief via Bayes rule to compute $c^*$, they use their knowledge of the equilibrium price function $P$.

Since $(\delta(\mu), \varepsilon(\mu))$ is sufficient for forecasting $\theta$ given $\mu$, the distribution of agents' individual demands depends only on $\delta$ and $\varepsilon$. The same is true for the aggregate excess demand at some price $p$. It follows that the equilibrium price can also be represented as a function $\tilde{P}$ of $\delta$ and $\varepsilon$. Below we will thus sometimes write $P(\mu) = \tilde{P}(\delta(\mu), \varepsilon(\mu))$. This perspective illustrates why a fully informative equilibrium—in which agents' know the distribution of payoffs $\tau$ given $\mu$—need not exist in our model. A fully informative equilibrium requires that agents can infer $\delta(\mu)$ from observing only the price and their own type. However, the unobservable distribution $\mu$ not only determines $\delta(\mu)$ but also the second parameter $\varepsilon(\mu)$, while there is only one price signal $\tilde{P}$. In general, inferring $\delta$ from $P$ need not be possible.

# 4 Exposure and the revelation of information

An important concept for characterizing trading and prices is investors' *exposure* to the aggregate risk factor $\tau$. We now describe a measure of exposure that can be applied to individual investors as well as to the economy as a whole. We then describe properties of equilibrium in terms of investors sharing exposure.

*Individual exposure*

Intuitively, an investor who is more exposed to a risk factor would pay more to reduce factor risk. We thus measure exposure of an investor holding portfolio $c$ to the risk factor $\tau$ by the premium this investor is willing to pay to avoid a marginal increase in $\tau$-risk. Consider increasing $\tau$-risk by $\Delta$ units of variance. For an investor with portfolio $c$, information set $I$ and utility $U$ defined on the set of random payoffs $C$, the risk premium $\rho(\Delta; c, I, U)$ per unit of variance is implicitly defined by

$$U(c - \Delta\rho(\Delta; c, I, U)) = U\left(c + \Delta\frac{\tau - E[\tau|I]}{var(\tau|I)}\right)$$

Here the left-hand side is utility at the portfolio $c$ less the sure payment of the risk premium, and the right-hand side is utility at $c$ plus an increase in risk. The increase in risk is described by a mean zero shock that is perfectly correlated with $\tau$ and scaled by $\Delta$ units of variance. Measuring exposure as a risk premium is reminiscent of familiar measures of risk aversion. The key difference is that we are not considering the premium for an increase in risk at certainty, which is always positive for a risk-averse investor. Instead, we are interested in premia for an increase in risk at a portfolio $c$ that is already risky. Such premia can be positive or negative even though the investor is risk-averse, depending on how the portfolio $c$ comoves with $\tau$.

We now define exposure as the premium for a *marginal* change in risk at the point $c = a_c + b_c\tau$. The limit of $\rho$ as $\Delta$ becomes small is computed by performing a Taylor expansion around $c$ and dividing by $\Delta$. Terms of order two and higher then vanish and we are left with

$$e(c,U) := \lim_{\Delta \to 0} \rho(\Delta; c, I, U) = \frac{1}{var(\tau|I)} \left( E[\tau|I] - \frac{dU/db_c}{dU/da_c} \right), \tag{5}$$

where all derivatives are evaluated at the point $c$. Exposure can be positive or negative depending on whether the investor's expectation of $\tau$ is larger or smaller than the marginal rate of substitution of $\tau$-risk for certain consumption. It is a subjective concept—all moments depend on the investor's subjective distribution given the information set $I$.

If $U$ has an expected utility representation $U(c) = E[u(c)|I]$ for some information set $I$ and felicity $u$, we can solve out the derivatives and simplify to obtain

$$e(c,U,I) = \frac{cov\left(\tau, -\frac{u'(c)}{E[u'(c)|I]}|I\right)}{var(\tau|I)}. \tag{6}$$

Exposure thus depends not only on the subjective distribution of the portfolio $c$ and the risk factor $\tau$, but also on the investor's utility function. On the one hand, the comovement of $\tau$ and $c$ determines the sign of $e$. Exposure is positive if and only if consumption covaries positively with $\tau$. Exposure is zero if consumption is independent of $\tau$. On the other hand, for a given distribution, exposure is lower if the investor is more risk tolerant—it is zero if the investor is risk neutral.

To continue our example of financial institutions trading bonds, a bank could have higher exposure to the housing market for one of two reasons. First, its bond portfolio could covary more strongly with the housing market, perhaps because it holds more subprime bonds with higher default rates. Second, its utility function could display higher risk aversion, for example because it is smaller and thus expects not to be covered by a too-big-to-fail policy. At the same time, size as such need not increase exposure and will leave it unaffected if utility is homothetic. Moreover, exposure is distinct from risk: An increase in the volatility of $\tau$ need not change exposure and will leave it unaffected for example if marginal utility is linear.

*Initial exposure and trading to optimal exposure under expected utility*

Our model describes how heterogenous investors trade to change their exposure in response to shocks. An important benchmark is an investor's *initial exposure*, defined as exposure at the endowment $\omega$ and in the absence of any signal about $\tau$. In other words, an investor is told the coefficients $a_\omega$ and $b_\omega$ that determine his initial portfolio, and then uses unconditional moments in (6) to compute

$$e(\omega; U, \varnothing) = \frac{cov\left(\tau, -\frac{u'(a_\omega + b_\omega \tau)}{E[u'(a_\omega + b_\omega \tau)]}\right)}{var(\tau)}.$$

For example, an institution with high initial exposure to the housing market may be one with a bond portfolio that on average (across shocks $\mu$) will comove a lot with housing, perhaps because the bank holds a lot of subprime bonds.

Consider now how an investor reacts to learning his type $\theta$ as well as the price $p$. Given our measure of individual exposure, we can rewrite the investor first order condition as

$$p = E\left[\tau | I(\theta)\right] - e\left(c^*(\theta; p); U(.;\theta), I(\theta)\right)\ var\left(\tau | I(\theta)\right) \tag{7}$$

Intuitively, an agent who is more optimistic about $\tau$ (has a higher $E\left[\tau | I(\theta)\right]$) and more confident about $\tau$ (lower $var(\tau)$) chooses to be more exposed to $\tau$. Equilibrium gives rise to a distribution of beliefs and exposures. In particular, in an equilibrium in which all investors have the same information (and hence the same beliefs about $E(\tau)$ and $var(\tau)$), they equate their exposures to the risk factor $\tau$ in equilibrium. Indeed if the mean and variance do not depend on $I(\theta)$, then neither can exposure. More generally, more optimistic and confident agents take on more $\tau$-risk than other agents. For example, more optimistic and confident institutions tilt their bond portfolios more toward subprime.[10]

*Aggregate exposure and the information revealed by prices*

Consider equilibria in which information aggregation by markets "works well." We say that an equilibrium is *fully informative* if the price is a sufficient statistic for forecasting the risk factor $\tau$ given $\mu$, the pooled information available at date 1. In other words, the price conveys the sufficient statistic $\delta(\mu)$. Investors then have common beliefs about $\tau$ in equilibrium: They all observe the price, and there is no value in looking at anything else. The equilibrium is the same as if the state $\mu$ were public information.

Common beliefs together with the fact that all risk is tradable implies the existence of a representative agent whose first order conditions can be used to price assets. Indeed, let $\lambda(\theta, \mu)$ denote the Lagrange multiplier on type $\theta$'s budget constraint in the aggregate state $\mu$, which is $P_{FI}$-measurable. We define the representative agent's utility on the consumption set $C$ by choosing $P_{FI}$-measurable consumption plans $(a, b)$ to solve

$$V(\Omega; \mu) := \max_{a,b} E\left[\lambda(\theta, \mu)^{-1} u\left(a(\theta) + b(\theta)\tau; \theta\right)\right]$$
$$\text{s.t. } a_\Omega + b_\Omega \tau = E\left[a(\theta)\right] + E\left[b(\theta)\right]\tau$$

[10]Trades that take investors from initial exposure to equilibrium exposure thus reflect two forces: updating of beliefs and equating exposure for the same beliefs.

Aggregate exposure can be defined analogously to individual exposure (5): It represents the risk premium that the representative agent would pay to avoid a marginal increase in risk at the aggregate endowment $\Omega$. This works even if the representative agent's utility function $V$ does not have an expected utility representation.

Using the representative agent's first order condition together with the general formula for exposure in (5), the price of a claim on the risk factor $\tau$ can be written as in (7), but now using the aggregate endowment and the information contained in $\mu$:

$$\begin{aligned} P_{FI}(\mu) &= \frac{dV/db_{\Omega}}{dV/da_{\Omega}} \\ &= E[\tau|\mu] - e(\Omega(\mu); V(.;\mu), \{\mu\}) \, var(\tau|\mu) \end{aligned} \tag{8}$$

On the second line, the first term is again simply the expected present value of payoffs (since the riskless interest rate does not appear when there is consumption at one date only). The second term is the risk premium written as the variance of $\tau$ multiplied by aggregate exposure to $\tau$. Risk matters for prices if and only if the representative agent is exposed to it. Other things equal, the price of a risk factor is lower the more the representative agent is exposed to that factor. This could be because the representative agent is more risk averse, or because the aggregate endowment comoves more with the risk factor.

A key property of fully informative equilibria is that *aggregate exposure is common knowledge*. Indeed, if agents know the conditional moments of $\tau$ given $\mu$, from their own signals $\theta$ as well as the price, then they must also know aggregate exposure. Markets thus aggregate information only if the distribution of types is sufficiently simple in the sense that agents know or learn aggregate exposure. For example, a fully informative equilibrium exists if aggregate exposure is independent of the parameter $\varepsilon(\mu)$, that is, it is either constant or perfectly correlated with $\delta$. More generally, shocks to the distribution $\varepsilon$ that move aggregate exposure will prevent equilibrium prices from revealing the relevant information. The rest of the paper considers properties of prices and trading in that case.

# 5 Uncertain exposure and asset pricing

In this section we work through a special case of the model aimed at deriving and illustrating, by means of simple graphs, our main results. The special case has two features. First, all investors have log utility: while this abstracts from heterogeneity in preferences, it does allow for wealth effects on portfolio choice and therefore effects of the wealth distribution on asset prices in an equilibrium with private information. Second, the risk factor $\tau$ takes values in the set $\{0, 1\}$. Its price $p$ is therefore the price of an Arrow security that pays off when $\tau = 1$.

Denote the prior probability of the event $\tau = 1$ by $\pi$. Evaluating (6) at the endowment yields initial exposure

$$e(\omega; U, \varnothing) = \frac{1}{a_\omega(\theta)/b_\omega(\theta) + 1 - \pi} \tag{9}$$

The ranking of investors by initial exposure is thus independent of the prior probability and depends only on the ratio of endowments in the two states, given by $1 + \frac{b_\omega(\theta)}{a_\omega(\theta)}$.

The distribution of types is set up so agents differ both by initial exposure and by information about $\tau$. Initial exposure depends on the endowment. There are two different endowment profiles, one with high exposure $\bar{e}$ and one with low exposure $\underline{e} < \bar{e}$. The parameter $\varepsilon$ represents the share of investors with high initial exposure; it takes a high value $\varepsilon^h$ with probability $\eta$ or a low value $\varepsilon^l < \varepsilon^h$ with probability $1 - \eta$. This setup implies that an investors' own initial exposure serves as a signal about the overall distribution of exposures. In particular, more exposed investors will believe that there are relatively more investors who also have high exposure.

The parameter $\delta$ represents the probability $\Pr(\tau = 1|\mu)$, which here summarizes the entire conditional distribution of $\tau$ given the pooled information $\mu$. It is drawn from a probability distribution $f(\delta|\varepsilon)$. Initial exposures may thus be correlated with the aggregate news $\delta$ and thereby with the risk factor $\tau$. News is dispersed in the economy in the form of private signals. In particular, an investor of type $\theta$ receives a private signal $s(\theta) \in \{s_1, s_0\}$ about the event $\tau = 1$. The signals $s(\theta)$ are iid across investors and independent of initial exposure. The probability of receiving a signal $s_1$ is equal to $\delta$. The signal realization $s_1$ is indicative of the risk factor realization $\tau = 1$—we will say that $s_1$ represents a "good" signal about $\tau$.

The aggregate endowment $\Omega(\mu)$ does not depend on the news $\delta$; we therefore directly write $\Omega(\varepsilon)$. With identical log utilities, the representative agent utility $V$ is also expected utility with log felicity. Aggregate exposure thus also takes the form (9):

$$e(\Omega(\varepsilon); V, \mu) = \frac{1}{a_{\Omega(\varepsilon)}/b_{\Omega(\varepsilon)} + 1 - \delta},$$

Conditionally on the aggregate news $\delta$, the aggregate exposure is strictly increasing $\varepsilon$. An increase in $\varepsilon$ thus corresponds to an increase in aggregate exposure.

*A numerical example*

To illustrate the main points of this section graphically, we fix a numerical example. It assumes that $\delta$ is drawn from a uniform distribution with support $[0,1]$, independently of $\varepsilon$, and $\eta = 0.5$. We also set $b(\theta) = 1$ for all investors. High exposure investors have $b(\theta) = 1$, low exposure investors have $b(\theta) = 0$, and $a(\theta) = 1$ for all types.

Agents with high initial exposure receive an endowment of $\omega_1(\theta)$ that is twice as large as the one received by agents with low exposure. The fraction of agents with high initial exposure either $\varepsilon^h = 0.9$ or $\varepsilon^l = 0.1$. As will be shown below, the model reduces to a system of nonlinear differential equations and we solve it numerically using Chebychev collocation. The code approximate the functions $\tilde{P}(\delta; \varepsilon_l)$ and $\tilde{P}(\delta; \varepsilon_h)$ as the weighted sum of fifteen Chebychev polynomials. The equilibrium is defined by the solution of a fixed point problem.

*Nonexistence of fully informative equilibria*

In a fully informative equilibrium, investors' equilibrium belief about $\tau$ is given by $\delta$. It is straightforward in this example, by using the consumer's first-order conditions, budget, and market clearing, to show that the equilibrium price must satisfy

$$\frac{\tilde{P}_{FI}(\delta, \varepsilon)}{1 - \tilde{P}_{FI}(\delta, \varepsilon)} = \frac{\delta}{1-\delta}\left(1 + \frac{b_{\Omega(\varepsilon)}}{a_{\Omega(\varepsilon)}}\right). \tag{10}$$

Figure 1 shows the equilibrium price function for the numerical example. The horizontal axis measures the news $\delta$. Any equilibrium is described by two curves, the price functions given low and high aggregate exposure, $\tilde{P}(., \varepsilon^l)$ and $\tilde{P}(., \varepsilon^h)$, respectively.

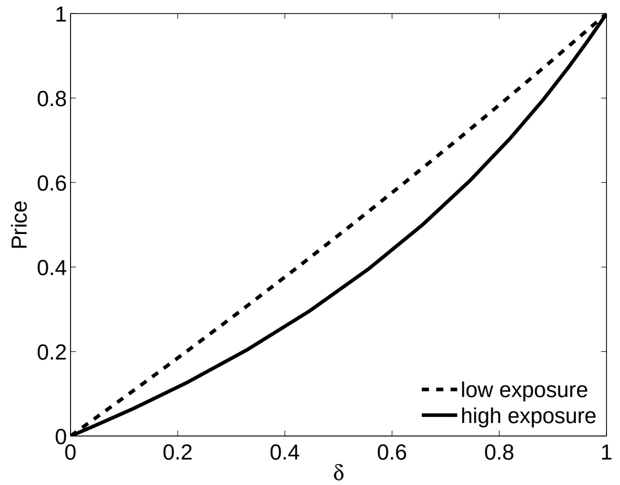


Figure 1: Prices under full information

Clearly, the full-information case has a zero price when $\delta = 0$ and a price of 1 for $\delta = 1$, it is increasing in $\delta$, it is decreasing in $\varepsilon$, and it is therefore nonlinear in $\delta$. It follows that a fully informative equilibrium cannot not exist when information is private and aggregate exposure depends on $\varepsilon$. Indeed, if there was such an equilibrium, agents would have to be able to infer $\delta$ from the price. However, for any given price, they cannot tell whether the price was generated by good news and low exposure, or by bad news and high exposure.

In equation (10), the relative price of the two contingent claims is conveniently expressed as containing two factors, one that depends only on $\delta$ and one that depends only on $\varepsilon$. It may not be apparent how this equation relates to the expression in the previous section, equation (8), where the price of the risk factor $\tau$ was written, also based on a first-order condition, as the expected value of $\tau$ less a term containing exposure times the variance of $\tau$. It is, however, easy to re-express (10) that way.[11] An advantage of the formulation in (10) is that the expected value of the risk factor $\tau$ and its variance—which both only depend only on $\delta$, the former equalling $\delta$ and the latter $\delta(1-\delta)$ in our example economy—appear jointly in the form of the ratio $\delta/(1-\delta)$.

*Equilibrium with private information*

Consider now equilibria that are not fully informative. If investors do not learn all the relevant information $\delta$ from prices, they base their forecast in part on their private signals. As a result, they disagree in equilibrium. Let $\hat{\delta}(\theta, p)$ denote type $\theta$'s subjective probability of the event $\tau = 1$ if the equilibrium price is $p$. Computation of $\hat{\delta}$ is by Bayes' rule, taking into account the investor's knowledge of the price function $\tilde{P}(\delta, \varepsilon)$ as well as the joint distribution of the state $(\delta, \varepsilon)$ and the private signals. Combining first order and market clearing conditions, we find that the equilibrium price must satisfy

$$\frac{\tilde{P}(\delta,\varepsilon)}{1-\tilde{P}(\delta,\varepsilon)} = \frac{\bar{\delta}_0(\delta,\varepsilon)}{1-\bar{\delta}_1(\delta,\varepsilon)}\left(1+\frac{b_{\Omega(\varepsilon)}}{a_{\Omega(\varepsilon)}}\right), \tag{11}$$

where $\bar{\delta}_\tau$ is an average of individual agents' beliefs $\hat{\delta}(\theta, p)$ weighted by agents' endowments conditional on the realization of $\tau$, that is,

$$\bar{\delta}_\tau(\delta,\varepsilon) = \sum_\theta \frac{\omega_\tau(\theta)}{\Omega_\tau(\varepsilon)}\hat{\delta}\left(\theta, \tilde{P}(\delta,\varepsilon)(\delta,\varepsilon)\right)\mu(\theta;\delta,\varepsilon).$$

[11] To see how, notice that the equivalent to equation (4) in the present setup is

$$\tilde{P}_{FI}(\delta,\varepsilon) = \delta - \delta(1-\delta)\frac{1}{1-\delta+\frac{a_{\Omega(\varepsilon)}}{b_{\Omega(\varepsilon)}}} = \underbrace{\delta}_{E[\tau|\mu]} - \underbrace{\delta(1-\delta)}_{var(\tau|\mu)}\underbrace{\frac{1}{1-\delta+\frac{a_{\Omega(\varepsilon)}}{b_{\Omega(\varepsilon)}}}}_{e(\Omega,u,\mu)}.$$

The representation (11) is similar to that in the full information case. In particular, there is the same direct negative effect of aggregate exposure on price. If all agents agree on $\delta$, we are thus back to (10). Here, though, because beliefs are heterogeneous, what matters in the price determination are two wealth-weighted averages of beliefs on $\delta$. Now there is consequently a key difference in how the price depends on news and aggregate exposure. On the one hand, the "true" unobservable news $\delta$ changes the price only to the extent that it shifts the mean of the (now nondegenerate) distribution of individual beliefs; actual aggregate exposure is not influenced by this news. On the other hand, aggregate exposure affects the distribution of beliefs and therefore affects prices also through a second, indirect channel. Finally, the beliefs $\hat{\delta}$ are described explicitly as a function of the price function so that it is clear how the equilibrium equation is a nontrivial functional equation.

The price function for the numerical example is shown as a pair of blue curves in Figure 2, where the full-information price function is also plotted.

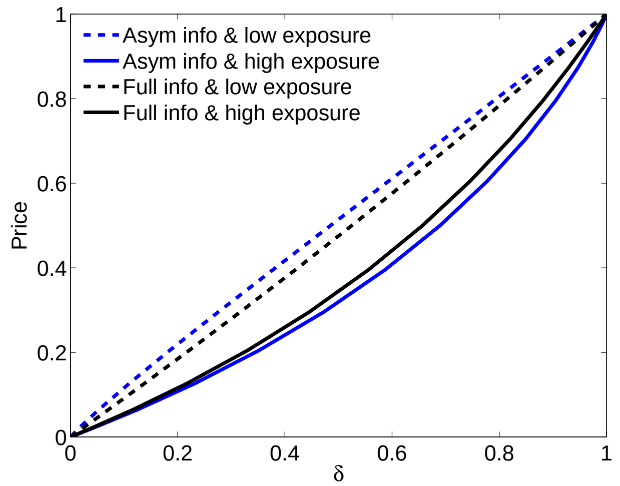


Figure 2: Prices under full and asymmetric information

The shape of the asymmetric-information price is overall similar: It is increasing in news $\delta$ and decreasing in exposure $\varepsilon$. In fact, holding fixed news, the effect of exposure on prices is relatively stronger in the economy with private information. To understand these effects, we now turn to analyzing inference from prices in equilibrium.

*Inference from prices and private information*

When investors look at the price, they think about whether it is driven by news or by aggregate exposure. For example, consider an investor who observes the price $p_0$ in the stylized Figure 3.

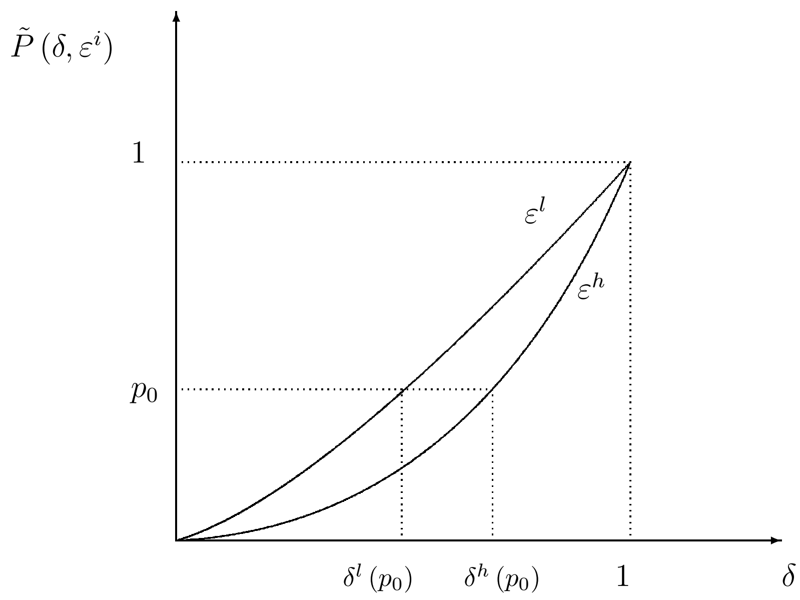


Figure 3: Signal extraction when agents observe a price p

The investor knows that only two values of $\delta$ could have been realized. Let $\delta^l(p_0)$ denote the value of aggregate news $\delta$ consistent with a price $p_0$ and low aggregate exposure. Similarly, let $\delta^h(p_0)$ denote the value of $\delta$ consistent with a price $p$ and high aggregate exposure. Since investors do not observe the actual distribution of individual exposures, they cannot distinguish which of the values $(\delta^l(p_0), \delta^h(p_0))$ corresponds to the actual realization of $\delta$.

Imperfect inference from prices implies that investor beliefs respond less to the true aggregate news. Indeed, Bayes' rule implies that each investor's belief consists of a weighted sum of $\delta^l(p_0)$ and $\delta^h(p_0)$). When aggregate exposure is high ($\delta = \delta^h(p_0)$), beliefs are then below the true $\delta$ realization as investors assign some weight to the possibility that $\delta = \delta^l(p_0)$. How an individual investor type weighs the role of news and exposure now depends on his private information, including his initial exposure.

A key implication is that investors with higher initial exposure are more optimistic in equilibrium. Indeed, for a given signal $s$, a high-exposure agent believes that it is more likely that the fraction of high-exposure agents is high rather than low, so he assigns more weight to $\delta^h(p_0)$ than a low-exposure agent. Similarly, for a given individual exposure, an agent with a good signal $s$ believes that it is more likely that the highest $\delta^i(p_0)$ was realized.

Given the four types, four possible beliefs about $\tau$ emerge in equilibrium, as shown for the numerical example in Figure 4. The beliefs of high-exposure types lie above the beliefs of low-exposure types. Similarly, the beliefs of agents with signals favorable to state $\tau = 1$ lie above the beliefs of agents with signals favorable to $\tau = 0$.

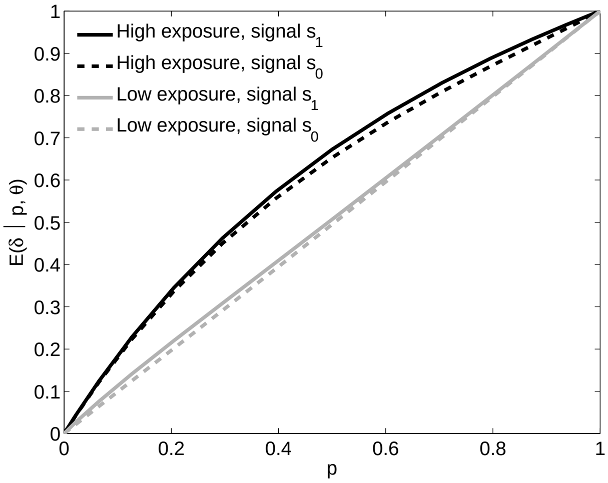


Figure 4: Equilibrium beliefs

The ranking of beliefs as well as the response of prices to aggregate exposure does not depends on the details of the numerical example. We summarize the discussion above in the following proposition, proved in the appendix.[12] To easily label the different types of agents, it is useful to write the type directly as $\theta = (s, e)$.

**Proposition 5.1.** *Consider an equilibrium with a price function $\tilde{P}$ that is continuous and increasing in $\delta$. Then*

1. *Individual beliefs are ranked by*

   (a) $\hat{\delta}(s, \bar{e}, p) > \hat{\delta}(s, \underline{e}, p)$ *(for a given signal, agents with higher exposure to the risk factor $\tau$ believe that $\tau$ is more likely to be high)*

[12]Formally, Bayesian updating yields that the posterior probability of the event $\tau = 1$ for an agent of type $\theta$ is

$$\hat{\delta}(\theta, p) = \hat{\eta}(\theta, p)\, \delta^h(p) + (1 - \hat{\eta}(\theta, p))\, \delta^l(p).$$

The term $\hat{\eta}(\theta, p)$ denotes the posterior probability that a type $\theta$ agent observing a price $p$ assigns to a distribution of types with high aggregate exposure $\mu = \mu_h(.|p)$, namely

$$\hat{\eta}(\theta, p) = \frac{\eta_p \mu_h(\theta; p)}{\eta_p \mu_h(\theta; p) + (1 - \eta_p)\, \mu_l(\theta; p)},$$

where $\eta_p$ denotes the probability that $\mu = \mu_h(.|p)$ based on the price alone, namely

$$\eta_p = \frac{\eta \left(\delta^h\right)'(p)\, f\left(\delta^h(p); \varepsilon^h\right)}{\eta \left(\delta^h\right)'(p)\, f\left(\delta^h(p); \varepsilon^h\right) + (1 - \eta)\left(\delta^l\right)'(p)\, f\left(\delta^l(p); \varepsilon^l\right)}. \quad (12)$$

The derivatives of the belief functions appear here, and since these derivatives are not constant due to the nonlinearity of the price function, the model solution amounts to the solution of a differential equation system.

(b) $\hat{\delta}(s_1, e, p) > \hat{\delta}(s_0, e, p)$ *(for given exposure, agents with a better signal about the risk factor $\tau$ believe that $\tau$ is more likely to be high).*

2. $\delta^h(p) > \delta^l(p)$ *(holding fixed the price, aggregate news about the risk factor $\tau$ is better when more agents have high exposure to $\tau$).*

Part 1.a of the proposition says that agents with higher exposure to the risk factor $\tau$ are more optimistic about the factor, that is, they believe that $\tau$ is more likely to take on the high value $\tau = 1$. Part 1.b simply says that, in a nonrevealing equilibrium, agents' beliefs also respond to their signals. This is because the signals add information over and above that contained in the price.

Part 2 of the proposition says that the news about $\tau = 1$ given the price and the number of agents with high initial exposure $\varepsilon$ must be better if $\varepsilon$ is higher. Part 2 also means that the price function $\tilde{P}$ is decreasing in $\varepsilon$. As in the full information case, higher aggregate exposure thus lowers prices.

*Private information and price volatility*

We have seen above how the presence of private information not only changes investors' inference of aggregate shocks from prices. We now consider how this change in inference feeds back to the effect of shocks on prices. The relevant comparison here is how the full information price function compares with the price function under private information. As Figure 2 illustrated for our numerical example, prices depend relatively more on aggregate exposure under private information. Formally, we have

**Proposition 5.2.** *Consider a nonrevealing equilibrium with price function $\tilde{P}$ that is continuous and strictly increasing in $\delta$. The equilibrium price depends more strongly on aggregate exposure than in the full information case: for every $\delta \in (0, 1)$,*

$$\tilde{P}\left(\delta, \varepsilon^l\right) > \tilde{P}_{FI}\left(\delta, \varepsilon^l\right) > \tilde{P}_{FI}\left(\delta, \varepsilon^h\right) > \tilde{P}\left(\delta, \varepsilon^h\right).$$

The intuition for this result can be seen in (11). The direct effect of aggregate exposure on price is the same regardless of the information structure. What is special under private information is that average beliefs also depend on aggregate exposure. In particular, when aggregate exposure is high, and prices are therefore low, then investors put some weight on the possibility that the price is low because of bad news. The average agent is thus more pessimistic than in the full information economy, which explains why prices are lower.

*Trading volume*

The ranking of beliefs translates directly into differences in trading behavior across agents. Indeed, compared with the full information case, equilibrium disagreement has two effects on trading volume that work in opposite directions. On the one hand, agents with the same endowments and preferences will choose different portfolios: Agents speculate based on their private signals. This effect tends to increase trading volume.

On the other hand, agents with different initial exposures to $\tau$ will not equate their exposures as they would in a full information equilibrium. Instead, precisely those agents who start with higher exposure end up more optimistic and hold on to their exposure. They do not receive as much insurance from agents with low exposure, because the latter are pessimistic about claims on $\tau$. This effect tends to lower trading volume relative to the full information case.

Figure 5 shows the effect of information on trading volume in the numerical example. Volume is measured as the market value of trades in the contingent claim paying out in the good state, as a function of price averaged across the two exposure outcomes.[13] In the asymmetric-information economy, the correlation between individual exposure and beliefs implies that there is less trading and, therefore, less sharing of exposure.

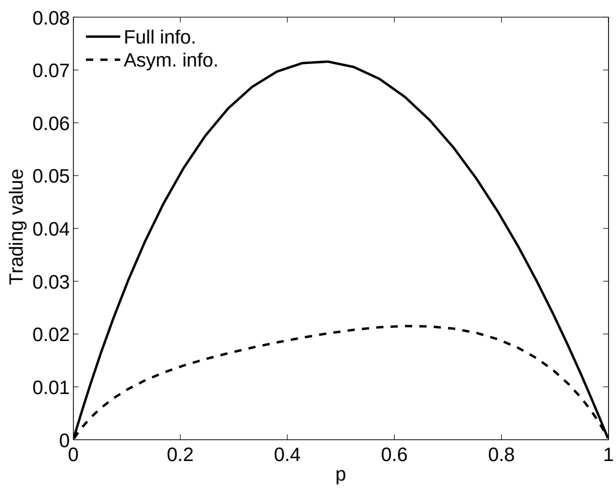


Figure 5: Trading volume

[13]Formally, the graph plots the aggregate trading value of claims paying on $\tau = 1$, i.e., $\sum_{j=l,h} Pr(\varepsilon^j) \sum_\theta \mu(\theta; \varepsilon^j, p)\, p |c_1(\theta; p) - \omega_1(\theta)|$, where $\mu(\theta; \varepsilon^j, p)$ denotes the distribution of types that generates a price $p$ when aggregate exposure takes the value $\varepsilon^j$, and $c_1(\theta; p)$ denotes the optimal consumption in state $\tau = 1$ of type $\theta$ when the price is $p$.

## 5.1 Excess volatility of asset prices

We have seen that shocks to aggregate exposure lower prices. We now ask whether there is a sense in which prices will appear "excessively low" in response to exposure shocks. One simple way to express this is to ask whether subsequent price changes are predictably positive, relative to a benchmark present value model. In other words, we ask whether excess returns on the risk factor are predictable. The excess return on a contingent claim that pays one unit when $\tau = \tau_1$ and zero otherwise is $r^e = \tau - p$.

Suppose an econometrician sees many realizations of excess returns and prices generated from the numerical example. On these data, he runs a regression of $x$ on the price of the contingent claim:

$$r^e = \alpha + \beta p + \nu,$$

where $\nu$ is an error term.[14]

| | Asym. info. | Full info. |
|---|---|---|
| $\alpha$ | 0.073 | 0.069 |
| $\beta$ | -0.031 | -0.017 |
| $R^2$ | 0.105 | 0.033 |

Table 1: Regressions of excess returns on prices

Table 1 presents summary statistics of the regression coefficients that the econometrician would obtain of the data were generated in an economy with asymmetric information or with full information. There is a negative relationship in both cases. Moreover, that relationship becomes more pronounced in the economy with private information. In addition, the R-squared is higher in the economy with asymmetric information.

Intuitively, predictability of excess returns requires shocks that affect the price without affecting the conditional expectation of the payoff conditional on the price. In our setup, changes in $\varepsilon$—shocks that affect aggregate exposure $\varepsilon$ but not aggregate news—serve this purpose. Exposure shocks are present in both economies. However, Proposition 5.2 says that they have a larger effect on prices in the economy with asymmetric information. Figure 2 shows that the

[14]In our two-state setting, the value of $\beta$ is independent of the asset used to run the regression. If the econometrician used the contingent claim paying when $\tau = 1$ the only coefficient that would change is $\alpha$. Similarly, the econometrician could use an asset with some other payoff contingent on $\tau$.

larger sensitivity of prices to aggregate exposure shocks is more pronounced at "intermediate" price values. At these prices there is more uncertainty about the actual values of $\delta$ and, thus, the discrepancy between agents' beliefs and $\delta$ becomes larger.

Figure 6 further illustrates this point. It plots the derivative of $E\left[\delta|p\right]$ with respect to the price in the economies with private and full information: The lower this derivative is, the more prices must fluctuate relative to expected returns. The figure shows that there is an intermediate range of price values where the sensitivity of expected excess returns is lower in the asymmetric information case.

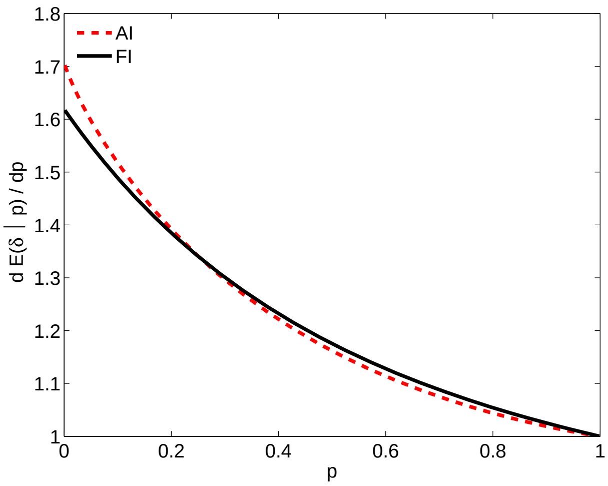


Figure 6: Sensitivity of expected returns to price changes, measured by the derivative $\frac{\partial E(\delta|p)}{\partial p}$, in the economies with full and asymmetric information

*Why predictability? Beliefs vs risk attitude*

The presence of return predictability suggests an exploitable portfolio strategy based on public information: At low prices, borrow at the riskless rate and buy claims on $\tau$; at high prices, sell short claims on $\tau$ and invest in the riskless asset. It is interesting to ask why the rational agents in the model do not exploit this strategy. The first-order condition of type $\theta$ implies that the expected excess return perceived by type $\theta$ must be equal to the risk premium perceived by type $\theta$. We can therefore write the expected excess return conditional on public information—namely, the price—as

$$E\left(\tau|p\right) - p_\tau = E\left[\tau|p\right] - E\left[\tau|\theta\right] + var\left(\tau|\theta\right) e\left(\theta, c(\theta), \hat{\delta}\left(\theta, p\right)\right) \tag{13}$$

Here the difference in expectations—the first term on the right-hand side—is a measure of pessimism of a type $\theta$ agent. The second term is the subjective risk premium of a type $\theta$ agent—his subjective variance of $x$ multiplied by his exposure. Consider now a low price, where the expected excess return is positive. This can be consistent with optimal behavior for two reasons: The agent can be more pessimistic than what one would be on the basis of public information, or he could demand a risk premium because of the exposure he chooses in equilibrium.

In the equilibrium of our model, *both* reasons apply. High-exposure types demand a high risk premium to hold claims on $\tau$, as their consumption level is more volatile and correlated with the payoff of these assets. Yet they are also optimistic about event $\tau = 1$, which deters them from selling too many of these claims. For low-exposure types, the consumption allocation is less sensitive to the realization of $\tau$ or it is negatively correlated with the payoff of this contingent claim. This implies that they demand a lower risk premium. At the same time, they are more pessimistic. This deters them from buying too many of these contingent claims.

## 5.2 Risk premia and trading in financial crises

While our model is stylized, the basic mechanism—changes in aggregate exposure matter more with asymmetric information—may help understand the behavior of asset prices and asset trading in the recent U.S. financial crisis. It has been widely reported that the market for mortgage-backed securities "seized up," and that the trades that did take place were done at "fire sale prices." One explanation discussed in the press runs as follows. Securities are heterogeneous and there is a lemons problem: Market participants cannot tell the good securities from the bad. As a result, only bad securities are traded, at low prices. This story assumes that securities are valued at their expected payoffs. The price of traded securities does not appear low to an agent who knows their expected payoff. In other words, there is not an unusually high risk premium.

Our model points to a different mechanism that also generates low trading activity at low sale prices. However, it works via low risk premia driven by uncertain exposure to aggregate risk. It does not rely on idiosyncratic risk in mortgage-backed securities. Instead, we think of two standardized securities. "Top-rated" mortgage-backed securities are riskless claims. "Junk" securities are bets on a risk factor $\tau$ which affects repayment on mortgages, such as house prices or economic activity. The agents in the model can be thought of as financial institutions, with risk aversion taken to be a stand-in for some imperfection (for example, risk aversion of undiversified managers, or an upward sloping cost of external finance).

Banks' exposure to mortgage risk depends on how many top-rated versus junk securities they have in their portfolios. Consider first an initial scenario, which may capture the situation before summer 2007. There are relatively few junk securities, and the exposure of the financial system to the risk factor is perceived to be small. Formally, using our two-state setting for purposes of illustration, the number of banks with high initial exposure is at the low value $\varepsilon^l$ and everybody knows this. Banks are therefore able to efficiently spread around exposure among themselves; for example banks that have originated subprime mortgages and have a high initial exposure to $\tau$ are able to package them into junk securities and sell them to other banks at relatively low risk premia reflected in the price $\tilde{P}_{FI}\left(\delta, \varepsilon^l\right)$.

Consider next a second scenario where the risk assessment of some top-rated securities has changed. Suddenly, many securities that were previously considered top-rated are no longer considered riskless. In terms of the model, assume that we move to a situation were some top-rated securities held by some banks are converted to junk. We thus consider the comparative static whereby the number of banks with high initial exposure increases to the high value $\varepsilon^h$. If the new distribution of exposure were known then the shock would lower the prices to $\tilde{P}_{FI}\left(\delta, \varepsilon^h\right)$ as risk premia increase, but should also lead to efficient sharing of exposure as banks who had a lot of top-rated securities turn to junk sell some of their junk to other, less exposed banks.

Assume now, however, that exposures are uncertain: Nobody knows precisely which banks and how many banks altogether have become more exposed. Formally, we consider the asymmetric information economy, where agents do not know whether the aggregate exposure is $\varepsilon^l$ or $\varepsilon^h$ (but the true exposure is $\varepsilon^h$). In addition, they do not know the true distribution $\delta$ of the risk factor $\tau$ that governs mortgage losses. Banks only see their own exposure, but need to estimate the aggregate exposure of the whole financial system, as well as the expected losses. Since banks know that everyone used similar risk assessment tools in the past, they take their own exposure as a signal of aggregate exposure. In addition, they observe the low price $\tilde{P}\left(\delta, \varepsilon^h\right)$.

Banks with high exposure believe that many other banks are similarly exposed. They therefore perceive the low price as largely due to an increase in aggregate exposure, rather than an increase in default probabilities (lower $\delta$). In contrast, banks with low exposure believe that the overall exposure of the financial system is low. They conclude that the low price must be reflecting higher default probabilities. As a result, they hesitate in purchasing securities from the high exposure banks, who hold on to their exposures.

At the same time, from the perspective of an observer, the price $\tilde{P}\left(\delta, \varepsilon^h\right)$—and therefore the price on any security that loads on the factor $\tau$—looks "too low,"like a "fire sale price."Indeed, consider an observer who has a good estimate of the actual expected payoff on the junk securities, or equivalently the true $\delta$, and who suspects that exposure might have increased. This observer knows that higher exposure will imply a higher risk premium and hence a lower price. From experience, he knows the size of risk premia in times when banks know aggregate exposure. He can thus compute the price that would obtain, in his experience, in the "worst case" state for aggregate exposure, namely $\tilde{P}_{FI}\left(\delta, \varepsilon^h\right)$. Comparing this price to the observed price $\tilde{P}\left(\delta, \varepsilon^h\right)$, the observer will then be puzzled to find that the market price on assets that depend on $\tau$ is even lower, and their risk premium even higher.

## 5.3 Wealth effects in asset pricing

In section 5.1 we have considered the excess volatility of asset prices relative to a present value model. Another interesting question is whether prices will appear excessively volatile relative to a benchmark model that allow for risk adjustment. One such benchmark is risk adjustment using total wealth, or equivalently here aggregate consumption.

We thus consider an econometrician who studies the Euler equation of a representative agent. The motivation comes from results in the asset pricing literature that suggest high and time varying risk aversion (that is, higher risk aversion when asset prices are low) can reconcile representative-agent models with the data. We ask whether the presence of private information can help understand these results.

Consider an econometrician who assumes that the data generating process comes from a representative-agent model with logarithmic preferences. He observes the joint distribution of $(\tau, \Omega, p)$: Asset payoffs, aggregate consumption, and the price. The econometrician does not know a priori the investors' information structure. He is aware of this, and therefore estimates the model by maximum likelihood, allowing for prices to depend on signals about future aggregate consumption and asset payoffs that agents receive at date 1.

An unrestricted estimation will recover the true joint distribution, summarized by the number $\eta$, the distribution of $\delta$, and the price function. In particular, the econometrician will find that movements in $\varepsilon$ (i.e., changes in aggregate consumption that are not in stock payoffs) are reflected in the price. He infers from this that the representative agent receives a signal that reveals $\varepsilon$.

However, when the econometrician imposes the cross-equation restrictions implied by log preferences, he will reject the model. Satisfying the cross-equation restrictions would require that, for all $p$,

$$\frac{p}{1-p} = \frac{\delta^j(p)}{1-\delta^j(p)} \frac{\Omega_0(\varepsilon^j)}{\Omega_1(\varepsilon^j)} \quad \text{for } j = h, l.$$

Equation 11 implies that this condition is typically violated.

To fix this problem, the econometrician can introduce preference shocks to fit the data exactly using his representative agent model. We capture the preference shock by specifying subjective beliefs $\tilde{\delta}^j(p)$ which depend on the price as well as on the state $j$. The econometrician thus determines $\tilde{\delta}^j(p)$ such that

$$\frac{p}{1-p} = \frac{\tilde{\delta}^j(p)}{1-\tilde{\delta}^j(p)} \frac{\Omega_0(\varepsilon^j)}{\Omega_1(\varepsilon^j)} \quad \text{for } j = h, l.$$

Proposition 5.1 now implies that $\tilde{\delta}^h(p) > \delta^h(p)$ and $\tilde{\delta}^l(p) < \delta^l(p)$. In other words, the econometrician's model will make agents more optimistic about $\tau$ in times of high aggregate exposure to $\tau$, and more pessimistic in times of low aggregate exposure. Since the price is decreasing in exposure, the econometrician has thus introduced a force that induces additional pessimism at low prices and optimism at high prices.

Of course, the price also depends on $\delta$, so we do not yet know whether the econometrician will conclude that the agent is pessimistic on average. The following proposition considers the econometrician's belief conditional on the price. It shows that the econometrician concludes agents are pessimistic at a price $p$ if aggregate wealth $W(\varepsilon, p) = p\Omega_1(\varepsilon) + (1-p)\Omega_0(\varepsilon)$ is positively correlated with aggregate exposure conditional on the price.

**Proposition 5.3.** *The econometrician's belief is more pessimistic conditional on the price if and only if*

$$W(\varepsilon^l, p) > W(\varepsilon^h, p),$$

*that is, there is more wealth in states with less aggregate exposure.*

The condition in Proposition 5.1 depends on the endogenous price $p$. We have $W(\varepsilon^l, p) > W(\varepsilon^h, p)$ if and only if

$$p\left(\Omega_1(\varepsilon^l) - \Omega_1(\varepsilon^h)\right) + (1-p)\left(\Omega_0(\varepsilon^l) - \Omega_0(\varepsilon^h)\right) > 0 \tag{14}$$

If moreover the aggregate endowment vectors are clearly ranked

$$\Omega_\tau(\varepsilon^l) > \Omega_\tau(\varepsilon^h), \qquad \tau = 0, 1 \tag{15}$$

then (14) holds for all values of $p \in (0, 1)$. Therefore, the econometrician will conclude that the agent is more pessimistic than what the date warrants given any price.

To sum up, suppose data are generated by an economy with log investors with rational expectations, where (15) holds. An econometrician who observes the data and studies the Euler equation of a log representative agent will reject the model. In particular, he will conclude that the agent is "too pessimistic."In other words, he will discover an equity premium puzzle—exogenously assumed pessimism has the same effect on unconditional moments as has higher risk aversion. Moreover, he will discover a force that increases risk aversion at low prices and vice versa. These findings do not reflect preferences with time varying risk aversion, but instead the econometrician's mistaken assumption that agents have symmetric information, so that standard representative-agent analysis applies.

# 6 Extensions

In this section, we consider several extensions that show how the basic themes of the paper work outside the concrete example of Section 3. We first point out that our main results do not depend on identical log utilities but carry over unchanged to heterogeneous preferences as long as all belong to the linear risk tolerance class. We then present an example to show that the basic logic also works in a setup with minimal assumptions on utility. Finally, relate our results to the commonly used framework with exponential utility and normal shocks. We show that the main results concerning the excess sensitivity of prices to aggregate exposure shocks and the adverse effect of uncertain aggregate exposure on trading volumes are also present in that setup.

## 6.1 Linear risk tolerance

For interpreting trading by financial institutions it is interesting to allow for differences in preferences across investors. This is because curvature in objective functions, for example due to changes in financing constraints, is likely to be a relevant source of heterogeneity in the data. Suppose that preferences are assumed to belong to the LRT class, with marginal risk tolerance equalized across agents. Formally, felicities are

$$u(c;\theta) = \begin{cases} \frac{\sigma}{\sigma-1}\left(\alpha(\theta)+\sigma c\right)^{1-\frac{1}{\sigma}} & \text{if } \sigma \notin \{0,1\} \\ \log\left(\alpha(\theta)+c\right) & \text{if } \sigma = 1 \\ -\alpha(\theta)\exp\left(-c/\alpha(\theta)\right) & \text{if } \sigma = 0 \text{ and } \alpha(\theta) > 0 \end{cases}$$

The common denominator of these preferences is that risk tolerance $-u'/u''$ (the inverse of the coefficient of absolute risk aversion) is given by the linear function $\alpha(\theta) + \sigma c$. Important special cases of LRT preferences are CRRA utility ($\alpha(\theta) = 0$, with $1/\sigma > 0$ the coefficient of relative risk aversion), CARA utility ($\sigma = 0$, with $\alpha(\theta) > 0$ the coefficient of absolute risk aversion), and quadratic utility ($\sigma = -1$). We require that the coefficient of marginal risk tolerance $\sigma$ be equal across agents. However, there can be differences in risk attitude independent of income that are captured by differences in $\alpha(\theta)$. For the case $\sigma > 0$, an intuitive way to think about the coefficient $\alpha(\theta)$ in our context is as a riskless endowment that cannot be traded away.

We show in the appendix that propositions 5.1 and 5.2 carry over without any modification. That is, the results that prices overreact to aggregate exposure shocks and that high-exposure types are more optimistic than low-exposure types do not depend on the logarithmic utility assumption

## 6.2 More general preferences

In the appendix A.3 we provide an example that is minimal in terms of the shock structure—there are only two aggregate states, so $\delta$ and $\varepsilon$ are perfectly correlated—but we make no assumptions on preferences utility beyond expected utility. We show conditions for the existence of an equilibrium with asymmetric information and that the main property underlying our results—investors with more exposure to a risk factor are more optimistic about the risk factor in the presence of private information—is present also in this setting. This shows that our main results do not hinge on preferences with LRT.

## 6.3 Exponential utility and normally distributed shocks

Here we consider a version of the model in which agents have exponential utility and face normally distributed shocks. We show that the excess sensitivity of prices to aggregate exposure shocks and the relationship between idiosyncratic exposure shocks and beliefs that we find in the discrete state setup also apply to this environment. The tradable aggregate risk factor $\tau$ realized at date 2 is now

$$\tau = \delta + w,$$

where $\delta$ and $w$ are independent and normally distributed with mean zero and variances $1/\pi_\delta$ and $1/\pi_w$, respectively. Here, thus, $\delta$ will play the same role as before—it will serve the role of an aggregate signal on the realization of the risk factor $\tau$—and $w$ is the shock that makes $\delta$ less than a perfect signal.

*Consumption set, types, and exposure*

Note that the assumption that agents choose normally distributed consumption plans from the set $C = \{c : \text{there are } a_c, b_c \in \mathbb{R} \text{ s.t. } c = a_c + b_c\tau\}$ is not restrictive: The optimal consumption that agents can achieve by trading a full range of claims contingent on $\tau$ belongs to $C$. The value of a consumption bundle in $C$ is $P(c) = a_c + b_c p$, where $p$ is a parameter of the price function. As in our two-state model, finding the equilibrium price for a given aggregate state boils down to finding one number $p$. Here it can be interpreted as the relative price of a claim on the factor relative to the price of a riskless asset.

The endowment of a type $\theta$ agent is the random variable $\omega(\theta) = a_\omega + b_\omega(\theta)\tau$. The agent's type thus determines the loading $b_\omega(\theta)$ of his endowment on the risk factor $\tau$, and it satisfies

$$b_\omega(\theta) = \varepsilon + v(\theta)$$

where $v(\theta)$ is normally distributed with mean zero and variance $1/\pi_v$. Thus, the loading has the component $\varepsilon$, which is aggregate and common across agents—it is aggregate exposure, in line with the notation in the general case above. We assume $\delta$ and $\varepsilon$ to be uncorrelated, as in the leading example in Section 4.[15]

The agent's type $\theta$ also determines an individual signal $s(\theta)$, which satisfies

$$s(\theta) = \delta + u(\theta).$$

Here, $u(\theta)$ is normally distributed with mean zero and variance $1/\pi_u$, and it is independent of all other random variables.

The preferences of all types are represented by exponential utility with coefficient of absolute risk aversion coefficient $\rho$, that is, $u(c) = -\exp(-\rho c)$. Assume that the belief of a type $\theta$ agent

[15]This assumption simplifies the algebra but is not essential.

about $\tau$ can be represented by a normal density $\hat{f}$. The exposure to the risk factor $\tau$ of a type $\theta$ agent with consumption plan $c = a_c + b_c\tau$ and belief $\hat{f}$ is defined as in (6), with moments evaluated using the density $\hat{f}$. It simplifies to

$$e\left(c, u, \hat{f}\right) = -\frac{cov\left(\tau, u'\left(c\right)\right)}{var\left(\tau|I\left(\theta\right)\right) E\left[u'\left(c\right)\right]} = b_c\rho,$$

and thus does not depend on $\hat{f}$.

As in the two-state example, exposure is positive if and only if the consumption plan is positively correlated with $\tau$, and exposure is zero if the consumption plan is independent of $\tau$ or the agent is risk neutral. The initial exposure of type $\theta$, defined using some arbitrary normal reference density $f$, is $e\left(\omega, u, f\right) = b_\omega\left(\theta\right)\rho$.[16] Since preferences are identical LRT preferences, there is a representative agent with the same utility function. The aggregate endowment is $\Omega\left(\varepsilon\right) = \bar{a}_\omega + \varepsilon\tau$, so aggregate exposure is given by $e\left(\Omega\left(\varepsilon\right), u, f\right) = \varepsilon\rho$.

*Results* It is straightforward, using standard signal-extraction techniques, to find individual decision rules and look for the market price function by solving a fixed-point problem. We have

**Proposition 6.4.** *In this economy,*

1. *there exists an equilibrium with a linear price function*

$$\tilde{P}\left(\delta, \varepsilon\right) = \beta\delta + \gamma\varepsilon.$$

2. *The price function is increasing in the aggregate dividend news $\delta$ (that is, $\beta > 0$) and decreasing in aggregate exposure $\varepsilon$ (that is, $\gamma < 0$).*

3. *Compared with the pooled information case, the price responds less to news and responds relatively more to aggregate exposure.*

4. *In equilibrium, agents with higher initial exposure are more optimistic about $\tau$, that is, the conditional expectation $E\left[\tau|\theta, p\right]$ is higher if the endowment load on $\tau$ ($b_\omega\left(\theta\right)$) is higher.*

Figure 7 provides an example in which trading volume is lower in the economy with private information.

The figure corresponds to an economy in which $\pi_\delta = \pi_w = 90$ and $\pi_\varepsilon = 400$. This implies a standard deviation of the aggregate risk factor of 14.9 percent (half of which could be learned

[16] Since the exposure measure does not depend on beliefs, there is no need for a second measure such as $\tilde{e}$ that we used in the two-state example above.

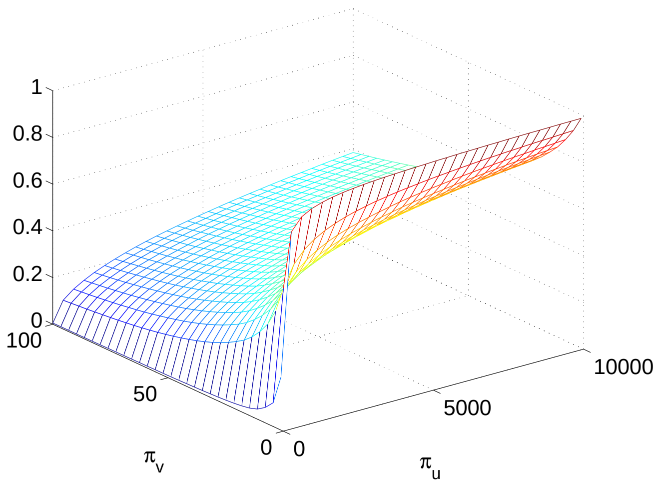


Figure 7: Average trading in economy with asymmetric information / average trading in economy with pooled information.

if agents pooled their information) and a standard deviation of the aggregate endowment of the risk factor of 5 percent. The figure, more specifically, shows the average trading volume in the asymmetric information economy relative to the one in the pooled information economy for different precisions of individual signals and idiosyncratic exposure shocks, namely

$$\frac{\int \mid \tilde{P}(\delta,\varepsilon)\left(b_c\left(\theta\right)-b_\omega\left(\theta\right)\right) \mid f_\delta(\delta)f_\varepsilon(\varepsilon)f_u(u)f_v(v)d\delta d\varepsilon dudv}{\int \mid \tilde{P^P}(\delta,\varepsilon)\left(b_c^P\left(\theta\right)-b_\omega\left(\theta\right)\right) \mid f_\delta(\delta)f_\varepsilon(\varepsilon)f_u(u)f_v(v)d\delta d\varepsilon dudv},$$

with $b_c^P = \frac{\delta - p}{\rho\pi_w^{-1}}$, and $\tilde{P^P} = \delta - \rho\pi_w^{-1}\varepsilon$ denoting the individual loadings on the risk factor and the price function in the pooled information economy, respectively.

The graph shows that average trading is lower than in the pooled information economy and that it decreases with the precision of the information about the news $\delta$.

# 7 Conclusions

We have argued that a Lucas-style model of asset pricing, augmented to allow for "speculative" trading motives due to differences in beliefs, can help us understand both asset prices and asset trading. In particular, we constructed a model where individuals have the same prior beliefs about asset payoffs but receive individual signals updating this belief, and due to signal extraction problems, they have different beliefs ex post. In our model, aggregate shocks to exposure play a prominent role: as in Lucas's model, exposure shocks influence prices, but under asymmetric information the impact of these shocks is stronger and it goes along with drops in trading volumes. We argued that these features can be used to interpret the recent financial crisis. We also demonstrated that the model's propagation to exposure shocks strengthens the predictability of asset returns, which helps understand the data.

It should be emphasized that the mechanisms we unveil here fundamentally rely on modeling belief formation as a rational (but subject to information dispersion and imperfect revelation by prices). Our characterization of who is pessimistic and who is optimistic, ex post, relies on Bayesian updating, and it lies behind the drop in volumes as well as the results on pricing. In contrast, a model with exogenous belief differences would, in general, be silent on these issues.

Imperfect revelation of aggregate information does rely on market incompleteness; in this sense, we rely on an exogenously assumed friction. The particular market incompleteness we consider is that there is no market in "exposure outcomes": aggregate exposure shocks constitute the noise that hinders agents from reading payoff news off of prices. It is perhaps comforting that the missing market here is not one that is directly payoff-relevant—exposure is, at least in our leading examples, independent of the asset payoff—but it would be important to further examine what deeper reasons could prevent such a market from emerging.

The use of our model for understanding extreme events may perhaps be extended to cover other cases, such as the episodes surrounding capital-market liberalizations. There, a key question is how exposure to the local market participants evolved after markets opened up. A lower exposure that is underestimated can, in this case, and for reasons parallel to those used for the crisis, lead to a very strong upward movement in the valuation of the local market—an over-reaction, because the price increase is partly misinterpreted as good payoff news. Whether the current sovereign debt crisis in Europe—in particular, the large drop in bond prices—can be better understood with the help of a model like that presented here, i.e., in terms of an imperfectly observed exposure shock, depends on the extent to which the traders in the market (large banks, etc.) were not sure of the distribution of the bonds across active traders. It is not clear to us to what extent this was an important factor.

The model has rich implications for welfare—both its ex-post distribution across agents and in ex-ante terms. Is there a role for government policy and, if so, what is it? In our competitive equilibrium, agents act optimally given prices, obviously, but can different actions (perhaps induced by policy) change the information transmission through prices and make risk sharing more efficient? These are issues well worth examining in detail, but they are beyond the scope of the present paper.

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# A Appendix: for online publication

## A.1 Proofs of propositions 5.1-5.3

This appendix collects all proofs. For the proofs for the two state setup, that is, propositions 5.1-5.3 and 6.1-6.3, the following notation is helpful. Define, for $j = h, l$, the functions $\tilde{P}^j : [0,1] \to [0,1]$ by $\tilde{P}^j(\delta) := \tilde{P}(\delta, \varepsilon^j)$. We focus on partially revealing equilibria in which the $\tilde{P}_j$ are continuous and strictly increasing in $\delta$. In this case, there are well-defined inverse functions defined by $\delta^j(p) = \tilde{P}_j^{-1}(p)$.

Consider the inference problem of a type $\theta$ agent. His posterior probability $\hat{\delta}(\theta, p)$ of the event $\tau = \tau_1$ is his posterior mean of the aggregate news $\delta$. If he observes a price $p$, knowledge of the price function tells him that the distribution $\mu$ is parameterized either by $(\delta^h(p), \varepsilon^h)$ or by $(\delta^l(p), \varepsilon^l)$. We denote the two distributions by $\mu_h(.;p)$ and $\mu_l(.;p)$, respectively. The probability that $\mu = \mu_h(.|p)$, based on the price alone, is

$$\eta_p = \frac{\eta \delta^{h\prime}(p) f\left(\delta^h(p); \varepsilon^h\right)}{\eta \delta_h'(p) f\left(\delta^h(p); \varepsilon^h\right) + (1-\eta)\delta^{l\prime}(p) f\left(\delta^l(p); \varepsilon^l\right)}, \tag{16}$$

where $f(\delta; \varepsilon$ denotes the density function of $\delta$ conditional on $\varepsilon$. In addition to the price, the agent observes his type, which is also a signal about the distribution $\mu$. Let $\hat{\eta}(\theta, p)$ denote the posterior probability that a type $\theta$ agent observing a price $p$ assigns to $\mu = \mu_h(.|p)$. It is given by

$$\hat{\eta}(\theta, p) = \frac{\eta_p \mu_h(\theta; p)}{\eta_p \mu_h(\theta; p) + (1-\eta_p)\mu_l(\theta; p)}.$$

An agent believes that the distribution $\mu_h$ is more likely if his own type $\theta$ is more likely to be drawn from $\mu_h$ relative to $\mu_l$. For example, a high exposure agent believes that it more likely that there are many high exposure agents. His posterior probability of the event $\tau = 1$ is then

$$\hat{\delta}(\theta, p) = \hat{\eta}(\theta, p)\delta^h(p) + (1 - \hat{\eta}(\theta, p))\delta^l(p).$$

**Proof of Proposition 5.1.**

As a preliminary step, we establish

**Lemma 1**. *(a)* $\hat{\delta}(s, \bar{e}, p) > \hat{\delta}(s, \underline{e}, p)$ *for all* $s$ *if and only if* $\delta^h(p) > \delta^l(p)$.
*(b)* $\hat{\delta}(s_1, e, p) > \hat{\delta}(s_0, e, p)$ *for all* $e$.

Proof. The individual belief $\hat{\delta}$ can be viewed as an average of $\delta^h$ and $\delta^l$,

$$\hat{\delta}(s, e, p) = \hat{\eta}(s, e, p)\delta^h(p) + (1 - \hat{\eta}(s, e, p))\delta^l(p) \tag{17}$$

where the individual weights are

$$\hat{\eta}(s,e,p) = \frac{\eta_p \mu_h(s,e)}{\eta_p \mu_h(s,e) + (1-\eta_p)\mu_l(s,e)}.$$

By independence of $s$ and $e$, the population weights are,

$$\mu_j(s_1,\bar{e};p) = \delta^j(p)\varepsilon^j$$

$$\mu_j(s_1,\underline{e};p) = \delta^j(p)\left(1-\varepsilon^j\right)$$

$$\mu_j(s_0,\bar{e};p) = \left(1-\delta^j(p)\right)\varepsilon^j$$

$$\mu_j(s_0,\underline{e};p) = \left(1-\delta^j(p)\right)\left(1-\varepsilon^j\right)$$

The implication (a) follows from the fact that $\hat{\eta}(s,\bar{e},p) > \hat{\eta}(s,\underline{e},p)$. Indeed, that statement is equivalent to

$$\frac{\mu_h(s,\bar{e};p)}{\mu_l(s,\bar{e};p)} > \frac{\mu_h(s,\underline{e};p)}{\mu_l(s,\underline{e};p)} \tag{18}$$

which is in turn equivalent to

$$\frac{\mu_h(s,\bar{e};p)}{\mu_h(s,\underline{e};p)} = \frac{\varepsilon^h}{1-\varepsilon^h} > \frac{\varepsilon^l}{1-\varepsilon^l} = \frac{\mu_l(s,\bar{e};p)}{\mu_l(s,\underline{e};p)}.$$

To show implication (b), consider first the case $\delta^h > \delta^l$. We want to show that $\hat{\eta}(s_1,e,p) > \hat{\eta}(s_0,e,p)$, which is equivalent to

$$\frac{\mu_h(s_1,e;p)}{\mu_l(s_1,e;p)} > \frac{\mu_h(s_0,e;p)}{\mu_l(s_0,e;p)}. \tag{19}$$

For any $e$, this is equivalent to

$$\frac{\delta^h}{\delta^l} > \frac{1-\delta^h}{1-\delta^l},$$

and thus holds if and only if $\delta^h > \delta^l$.

In the case $\delta^h < \delta^l$, we want to show that $\hat{\eta}(s_1,e,p) < \hat{\eta}(s_0,e,p)$, that is, the reverse of (19), which holds iff $\delta^h < \delta^l$.■

We now establish **Part 2** of the proposition. **Part 1** then follows immediately from Lemma 1

Start from the market clearing condition for the claim on $\tau = 1$:

$$p\sum_{\theta}\mu_j(\theta;p)\,c_1(\theta;\mu_j) = p\Omega_1\left(\varepsilon^j\right)$$

The first order conditions and budget constraint for agent $\theta$ can be written as

$$\frac{\hat{\delta}(\theta,p)}{1-\hat{\delta}(\theta,p)}\frac{c_0}{c_1} = \frac{p}{1-p}$$
$$pc_1 + (1-p)\,c_0 = w(\theta,p)$$

Combining the first order conditions and budget constraint, we can then rewrite the market clearing conditions as

$$\sum_{\theta} \mu_j(\theta;p)\,\hat{\delta}(\theta,p)\,w(\theta,p) = p\Omega_1\left(\varepsilon^j\right)$$

Intuitively, the function $\psi$ acts as an "adjusted" expenditure share in a world where endowments and consumption have been linearly translated using the parameters $\sigma$ and $\alpha$. Using the definition of wealth, we now have

$$\frac{p}{1-p} = \left(\frac{\bar{\delta}_0(\varepsilon^j,p)}{1-\bar{\delta}_1(\varepsilon^j,p)}\right)\left(\frac{\Omega_0(\varepsilon^j)}{\Omega_1(\varepsilon^j)}\right) \quad \text{for } j=h,l. \tag{20}$$

where

$$\bar{\delta}_\tau\left(\varepsilon^j,p\right) = \sum \mu_j(\theta)\,\frac{\omega_\tau(\theta)}{\Omega_\tau(\varepsilon^j)}\hat{\delta}(\theta,p) \quad \text{for } \tau=0,1$$

is the average of beliefs $\hat{\delta}$ weighted by endowments in states $\tau$ and $j$.

Now suppose towards a contradiction that $\delta^h(p) \leq \delta^l(p)$. By Lemma 1, the individual beliefs are ordered as $\hat{\delta}(s,\bar{e},p) \leq \hat{\delta}(s,\underline{e},p)$ and $\hat{\delta}(s_1,e,p) \geq \hat{\delta}(s_0,e,p)$.

The averages $\bar{\delta}_\tau(\varepsilon^j)$ can now be ranked, for $\tau=0,1$:

$$\begin{aligned}
\bar{\delta}_\tau\left(\varepsilon^h\right) &= \frac{\varepsilon^h\omega_\tau(\bar{e})}{\Omega_\tau(\varepsilon^h)}\left[\delta^h\hat{\delta}(s_1,\bar{e},p) + \left(1-\delta^h\right)\hat{\delta}(s_0,\bar{e},p)\right] \\
&+ \frac{\left(1-\varepsilon^h\right)\omega_\tau(\underline{e})}{\Omega_\tau(\varepsilon^h)}\left[\delta^h\hat{\delta}(s_1,\underline{e},p) + \left(1-\delta^h\right)\hat{\delta}(s_0,\underline{e},p)\right] \\
&\leq \frac{\varepsilon^l\omega_\tau(\bar{e})}{\Omega_\tau(\varepsilon^l)}\left[\delta^h\hat{\delta}(s_1,\bar{e},p) + \left(1-\delta^h\right)\hat{\delta}(s_0,\bar{e},p)\right] \\
&+ \frac{\left(1-\varepsilon^l\right)\omega_\tau(\underline{e})}{\Omega_\tau(\varepsilon^l)}\left[\delta^h\hat{\delta}(s_1,\underline{e},p) + \left(1-\delta^h\right)\hat{\delta}(s_0,\underline{e},p)\right] \\
&\leq \frac{\varepsilon^l\omega_\tau(\bar{e})}{\Omega_\tau(\varepsilon^l)}\left[\delta^l\hat{\delta}(s_1,\bar{e},p) + \left(1-\delta^l\right)\hat{\delta}(s_0,\bar{e},p)\right] \\
&+ \frac{\left(1-\varepsilon^l\right)\omega_\tau(\underline{e})}{\Omega_\tau(\varepsilon^l)}\left[\delta^l\hat{\delta}(s_1,\underline{e},p) + \left(1-\delta^l\right)\hat{\delta}(s_0,\underline{e},p)\right] \\
&= \bar{\delta}_\tau\left(\varepsilon^l\right),
\end{aligned} \tag{21}$$

where the first inequality holds because $\hat{\delta}(s,\bar{e},p) \leq \hat{\delta}(s,\underline{e},p)$ and $\varepsilon^h > \varepsilon^l$, and where the second inequality holds because $\hat{\delta}(s_1,e,p) \geq \hat{\delta}(s_0,e,p)$ and $\delta^h \leq \delta^l$.

Given that aggregate exposure is strictly increasing in $\varepsilon$, it thus follows that

$$\frac{\Omega_0\left(\varepsilon^l\right)}{\Omega_1\left(\varepsilon^l\right)} > \frac{\Omega_0\left(\varepsilon^h\right)}{\Omega_1\left(\varepsilon^h\right)} \tag{22}$$

Putting together inequalities (21) and (22), we have

$$\begin{aligned}\frac{p}{1-p} &= \frac{\bar{\delta}_0\left(\varepsilon^h\right)}{1-\bar{\delta}_1(\varepsilon^h)}\frac{\Omega_0\left(\varepsilon^h\right)}{\Omega_1\left(\varepsilon^h\right)} \\ &< \frac{\bar{\delta}_0\left(\varepsilon^l\right)}{1-\bar{\delta}_1(\varepsilon^l)}\frac{\Omega_0\left(\varepsilon^l\right)}{\Omega_1\left(\varepsilon^l\right)},\end{aligned}$$

which contradicts the equilibrium condition (20).

■

**Proof of Proposition 5.2.**

Using analogous notation as for the REE price function, we define $P_{FI}^j(\delta) := P_{FI}(\delta, \varepsilon^j)$. By (10) in the proof of Proposition 5.1, the functions $P_{FI}^j$ are strictly increasing and thus have well-defined inverse functions $\delta_{FI}^j(p) = \left(P_{FI}^j\right)^{-1}(p)$.

We now establish that for all $p \in (0,1)$,

$$\delta^h(p) > \delta_{FI}^h(p) > \delta_{FI}^l(p) > \delta^l(p).$$

Since both $\tilde{P}$ and $\tilde{P}_{FI}$ are strictly increasing and continuous in $\delta$, this proves part 3.

In the full information case, we can follow the same algebra as in the proof of Proposition 5.1. above to arrive at equation (20). If all beliefs are equal at $\hat{\delta}(\theta,p) = \delta^j(p)$, that equation simplifies to

$$\frac{p}{1-p} = \frac{\delta_{FI}^j(p)}{1-\delta_{FI}^j(p)}\frac{\Omega_0\left(\varepsilon^j\right)}{\Omega_1\left(\varepsilon^j\right)} \quad \text{for } j = h, l. \tag{23}$$

Since all the $\bar{\delta}_\tau(\varepsilon^j)$ are averages of the $\hat{\delta}(\theta,p)$, which are averages of $\delta^h(p)$ and $\delta^l(p)$, we have $\bar{\delta}_\tau\left(\varepsilon^h\right) < \max_\theta \hat{\delta}(\theta,p) < \delta^h(p)$ for $\tau = 0,1$. This and equation (20) imply

$$\frac{p}{1-p} < \frac{\delta^h(p)}{1-\delta^h(p)}\frac{\Omega_0\left(\varepsilon^h\right)}{\Omega_1\left(\varepsilon^h\right)}. \tag{24}$$

The definition of $\delta_{FI}^h$ in (23) thus implies $\delta_{FI}^h < \delta^h(p)$. The argument for $\delta_{FI}^l > \delta^l$ follows analogously from the fact that $\bar{\delta}_\tau\left(\delta, \varepsilon^l\right) > \delta^l(p)$ for $\tau = 1,2$.

■

**Proof of Proposition 5.3.** We want to show

$$\eta_p^h \hat{\delta} + (1-\eta_p)\,\hat{\delta}^l < \eta_p \delta^h + (1-\eta_p)\,\delta^l.$$

or equivalently

$$\eta_p \left(\hat{\delta}^h - \delta^h\right) + (1-\eta_p)\left(\hat{\delta}^l - \delta^l\right) < 0$$

Market clearing at the price $p$ is

$$\sum_\theta \mu^j(\theta)\,\hat{\delta}(\theta,p)\,w(\theta,p) = p\sum_\theta \mu^j(\theta)\,\omega_1(\theta)\,, \tag{25}$$

where

$$\begin{aligned}
\hat{\delta}(\theta,p) &= \hat{\eta}(\theta,p)\,\delta^h(p) + \left(1-\hat{\eta}(\theta,p)\right)\delta^l(p) \\
\hat{\eta}(\theta,p) &= \frac{\eta_p \mu_h(\theta)}{\eta_p \mu_h(\theta) + (1-\eta_p)\mu_l(\theta)} \\
\eta_p &= \frac{\eta \delta^{h\prime} f\left(\delta^h\right)}{\eta \delta^{h\prime} f\left(\delta^h\right) + (1-\eta)\,\delta^{l\prime} f\left(v_l\right)}
\end{aligned}$$

Multiply the market clearing equation for state $h$ by $\eta_p$, multiply that for state $l$ by $(1-\eta_p)$ and add the two equations to get

$$\begin{aligned}
&\eta_p \sum_\theta \mu^h(\theta)\,\frac{\eta_p \mu_h(\theta)\,\delta^h + (1-\eta_p)\,\mu_l(\theta)\,\delta^l}{\eta_p \mu_h(\theta) + (1-\eta_p)\mu_l(\theta)} w(\theta,p) \\
&+ (1-\eta_p)\sum_\theta \mu^l(\theta)\,\frac{\eta_p \mu_h(\theta)\,\delta^h + (1-\eta_p)\,\mu_l(\theta)\,\delta^l}{\eta_p \mu_h(\theta) + (1-\eta_p)\mu_l(\theta)} w(\theta,p) \\
&= p\left(\eta_p \sum_\theta \mu^h(\theta)\,\omega_1(\theta) + (1-\eta_p)\sum_\theta \mu^l(\theta)\,\omega_1(\theta)\right)
\end{aligned}$$

Rearranging terms we get

$$\eta_p \delta^h W\left(\varepsilon^h,p\right) + (1-\eta_p)\,\delta^l W\left(\varepsilon^l,p\right) = p\left(\eta_p \Omega_1\left(\varepsilon^h\right) + (1-\eta_p)\,\Omega_1\left(\varepsilon^l\right)\right), \tag{26}$$

where $W\left(\varepsilon^j,p\right)$ is aggregate wealth in state $j$.

Now the subjective beliefs fit by the econometrician satisfy

$$\hat{\delta}^j W\left(\varepsilon^j,p\right) = p\Omega_1\left(\varepsilon^j\right), \qquad j = h,l.$$

We can again multiply the equations for $h$ and $l$ by $\eta_p$ and $1-\eta_p$, respectively. We get that

$$\eta_p^h \hat{\delta}^h W\left(\varepsilon^h,p\right) + (1-\eta_p)\,\hat{\delta}^l W\left(\varepsilon^l,p\right) = p\left(\eta_p \Omega_1\left(\varepsilon^h\right) + (1-\eta_p)\,\Omega_1\left(\varepsilon^l\right)\right), \tag{27}$$

Combining (26) and (27), we have

$$\eta_p\left(\hat{\delta}^h - \delta^h\right) W\left(\varepsilon^h,p\right) + (1-\eta_p)\left(\hat{\delta}^l - \delta^l\right) W\left(\varepsilon^l,p\right) = 0$$

But then

$$\eta_p\left(\hat{\delta}^h-\delta^h\right)+(1-\eta_p)\left(\hat{\delta}^l-\delta^l\right)=\eta_p\left(\hat{\delta}^h-\delta^h\right)-(1-\eta_p)\frac{\eta_p\left(\hat{\delta}^h-\delta^h\right)W\left(\varepsilon^h,p\right)}{(1-\eta_p)\,W\left(\varepsilon^l,p\right)}$$
$$=\frac{\eta_p}{W\left(\varepsilon^l,p\right)}\left(\hat{\delta}^h-\delta^h\right)\left(W\left(\varepsilon^l,p\right)-W\left(\varepsilon^h,p\right)\right)$$

Since $\hat{\delta}^h<\delta^h$ from Proposition 5.2., the condition follows.■

## A.2 The Economy with Linear Absolute Risk Tolerance

This subsection presents propositions 6.1-6.2, which show that the main results obtained with logarithmic utility function carry over with preferences that exhibit LRT.

A convenient feature of the LRT family of preferences is that, when agents observe the pooled information and learn $\delta(\mu)$, there exists a representative agent who has an LRT felicity function with the same coefficient of marginal risk tolerance $\sigma$ as the individual agents and the average coefficient $\alpha$:

$$\sum_{\theta\in\Theta}\mu(\theta)\,\alpha(\theta)=:\bar{\alpha}\left(\varepsilon\left(\mu\right)\right).$$

Here $\bar{\alpha}$ is well defined as a function of $\varepsilon$ only, because we have assumed that types with the same initial exposure have the same felicity function.

For $\sigma\neq 0$, aggregate exposure is determined by

$$e\left(\Omega(\mu);u;\mu\right)=\frac{1}{1-\delta(\mu)+\frac{1}{\left(\frac{\bar{\alpha}(\varepsilon(\mu))+\sigma\Omega_1(\mu)}{\bar{\alpha}(\varepsilon(\mu))+\sigma\Omega_0(\mu)}\right)^{(1/\sigma)}-1}}.$$

and the price function $P_{FI}\left(\mu\right)=\tilde{P}_{FI}\left(\delta\left(\mu\right),\varepsilon\left(\mu\right)\right)$ (in the full information case) is available in closed form using equation (10). Propositions 6.1 and 6.2 show that the results presented in propositions 5.1 and 5.2 carry over when utility functions display linear risk tolerance.

**Proposition 6.1.** *Consider an equilibrium with a price function $\tilde{P}$ that is continuous and increasing in $\delta$. Then*

1. *Individual beliefs are ranked by*

   (a) $\hat{\delta}\left(s,\bar{e},p\right)>\hat{\delta}\left(s,\underline{e},p\right)$ *(for a given signal, agents with higher exposure to the risk factor $\tau$ believe that $\tau$ is more likely to be high)*

   (b) $\hat{\delta}\left(s_1,e,p\right)>\hat{\delta}\left(s_0,e,p\right)$ *(for given exposure, agents with a better signal about the risk factor $\tau$ believe that $\tau$ is more likely to be high).*

2. $\delta^{h}(p) > \delta^{l}(p)$ *(holding fixed the price, aggregate news about the risk factor $\tau$ is better when more agents have high exposure to $\tau$).*

**Proposition 6.2** *Consider a nonrevealing equilibrium with price function $\tilde{P}$ that is continuous and strictly increasing in $\delta$. The equilibrium price depends more strongly on aggregate exposure than in the full information case: for every $\delta \in (0,1)$,*

$$\tilde{P}\left(\delta, \varepsilon^{l}\right) > \tilde{P}_{FI}\left(\delta, \varepsilon^{l}\right) > \tilde{P}_{FI}\left(\delta, \varepsilon^{h}\right) > \tilde{P}\left(\delta, \varepsilon^{h}\right).$$

**Proof of Proposition 6.1.**

See proof of proposition 5.1 for **Part 1** . We now establish **Part 2** of the proposition. **Part 1** then follows immediately from Lemma 1

We begin with the case $\sigma \neq 0$. Start from the market clearing condition for the claim on $\tau = 1$:

$$p \sum_{\theta} \mu_{j}(\theta; p)\, c_{1}(\theta; \mu_{j}) = p\Omega_{1}\left(\varepsilon^{j}\right)$$

Multiplying the equation by $\sigma$, adding $\bar{\alpha}(\varepsilon^{j})$ and rearranging, we obtain

$$p \sum_{\theta} \mu_{j}(\theta; p)\left(\alpha(\theta) + \sigma c_{1}(\theta; \mu_{j})\right) = p\left(\bar{\alpha}\left(\varepsilon^{j}\right) + \sigma\Omega_{1}\left(\varepsilon^{j}\right)\right)$$

The first order conditions and budget constraint for agent $\theta$ can be written as

$$\frac{\hat{\delta}(\theta, p)}{1 - \hat{\delta}(\theta, p)} \left(\frac{\alpha(\theta) + \sigma c_{1}}{\alpha(\theta) + \sigma c_{0}}\right)^{-\frac{1}{\sigma}} = \frac{p}{1-p}$$

$$p\left(\alpha(\theta) + c_{1}\right) + (1-p)\left(\alpha(\theta) + c_{0}\right) = w(\theta, p) + \alpha(\theta)$$

Define the expenditure share in the case of power utility with belief $\delta$ by

$$\psi(\delta, p) := \frac{1}{1 + \left(\frac{1-p}{p}\right)^{1-\sigma}\left(\frac{1-\delta}{\delta}\right)^{\sigma}}.$$

Combining the first order conditions and the definition of $\psi$, we can then rewrite the market clearing conditions as

$$\sum_{\theta} \mu_{j}(\theta; p)\, \psi\left(\hat{\delta}(\theta, p), p\right)\left(\alpha(\theta) + \sigma w(\theta, p)\right) = p\left(\bar{\alpha}\left(\varepsilon^{j}\right) + \sigma\Omega_{1}\left(\varepsilon^{j}\right)\right)$$

Intuitively, the function $\psi$ acts as an "adjusted" expenditure share in a world where endowments and consumption have been linearly translated using the parameters $\sigma$ and $\alpha$. Using the definition of wealth, we now have

$$\frac{p}{1-p} = \left(\frac{\bar{\psi}_0\left(\varepsilon^j, p\right)}{1-\bar{\psi}_1(\varepsilon^j, p)}\right)\left(\frac{\bar{\alpha}\left(\varepsilon^j\right)+\sigma\Omega_0\left(\varepsilon^j\right)}{\bar{\alpha}\left(\varepsilon^j\right)+\sigma\Omega_1\left(\varepsilon^j\right)}\right) \quad \text{for } j = h, l. \tag{28}$$

where

$$\bar{\psi}_\tau\left(\varepsilon^j, p\right) = \sum \mu_j\left(\theta\right)\frac{\alpha\left(\theta\right)+\sigma\omega_\tau\left(\theta\right)}{\bar{\alpha}\left(\varepsilon^j\right)+\sigma\Omega_\tau\left(\varepsilon^j\right)}\psi\left(\hat{\delta}\left(\theta, p\right), p\right) \quad \text{for } \tau = 0, 1$$

is an average of the adjusted expenditure shares $\psi$ formed by weighting individual adjusted expenditure shares by adjusted endowments in states $\tau$ and $j$.

Now suppose towards a contradiction that $\delta^h\left(p\right) \leq \delta^l\left(p\right)$. By Lemma 1, the individual beliefs are ordered as $\hat{\delta}\left(s, \bar{e}, p\right) \leq \hat{\delta}\left(s, \underline{e}, p\right)$ and $\hat{\delta}\left(s_1, e, p\right) \geq \hat{\delta}\left(s_0, e, p\right)$.

The effect of beliefs on the adjusted expenditure shares $\psi$ depends on the sign of $\sigma$. We begin with the case $\sigma > 0$. To simplify notation, write $\hat{\psi}\left(e, s, p\right) := \psi\left(\hat{\delta}\left(s, \bar{e}, p\right), p\right)$. If $\sigma > 0$, the function $\psi$ is strictly increasing in $\delta$, which implies $\hat{\psi}\left(s, \bar{e}, p\right) \leq \hat{\psi}(s, \underline{e}, p)$ and $\hat{\psi}\left(s_1, e, p\right) \geq \hat{\psi}\left(s_0, e, p\right)$.

The averages $\bar{\psi}_\tau\left(\varepsilon^j\right)$ can now be ranked, for $\tau = 0, 1$:

$$\begin{aligned}
\bar{\psi}_\tau\left(\varepsilon^h\right) &= \frac{\varepsilon^h\omega_\tau\left(\bar{e}\right)}{\Omega_\tau\left(\varepsilon^h\right)}\left[\delta^h\hat{\psi}\left(s_1, \bar{e}, p\right)+\left(1-\delta^h\right)\hat{\psi}\left(s_0, \bar{e}, p\right)\right] \\
&+ \frac{\left(1-\varepsilon^h\right)\omega_\tau\left(\underline{e}\right)}{\Omega_\tau\left(\varepsilon^h\right)}\left[\delta^h\hat{\psi}\left(s_1, \underline{e}, p\right)+\left(1-\delta^h\right)\hat{\psi}\left(s_0, \underline{e}, p\right)\right] \\
&\leq \frac{\varepsilon^l\omega_\tau\left(\bar{e}\right)}{\Omega_\tau\left(\varepsilon^l\right)}\left[\delta^h\hat{\psi}\left(s_1, \bar{e}, p\right)+\left(1-\delta^h\right)\hat{\psi}\left(s_0, \bar{e}, p\right)\right] \\
&+ \frac{\left(1-\varepsilon^l\right)\omega_\tau\left(\underline{e}\right)}{\Omega_\tau\left(\varepsilon^l\right)}\left[\delta^h\hat{\psi}\left(s_1, \underline{e}, p\right)+\left(1-\delta^h\right)\hat{\psi}\left(s_0, \underline{e}, p\right)\right] \\
&\leq \frac{\varepsilon^l\omega_\tau\left(\bar{e}\right)}{\Omega_\tau\left(\varepsilon^l\right)}\left[\delta^l\hat{\psi}\left(s_1, \bar{e}, p\right)+\left(1-\delta^l\right)\hat{\psi}\left(s_0, \bar{e}, p\right)\right] \\
&+ \frac{\left(1-\varepsilon^l\right)\omega_\tau\left(\underline{e}\right)}{\Omega_\tau\left(\varepsilon^l\right)}\left[\delta^l\hat{\psi}\left(s_1, \underline{e}, p\right)+\left(1-\delta^l\right)\hat{\psi}\left(s_0, \underline{e}, p\right)\right] \\
&= \bar{\psi}_\tau\left(\varepsilon^l\right),
\end{aligned} \tag{29}$$

where the first inequality holds because $\hat{\psi}\left(s, \bar{e}, p\right) \leq \hat{\psi}(s, \underline{e}, p)$ and $\varepsilon^h > \varepsilon^l$, and where the second inequality holds because $\hat{\psi}\left(s_1, e, p\right) \geq \hat{\psi}\left(s_0, e, p\right)$ and $\delta^h \leq \delta^l$.

We also know that aggregate exposure is strictly increasing in $\varepsilon$.[17] If $\sigma > 0$, it thus follows that

[17] Recall that in this case aggregate exposure in state $\varepsilon^j$ is equal to

$$\frac{\bar{\alpha}\left(\varepsilon^{l}\right)+\sigma\Omega_{0}\left(\varepsilon^{l}\right)}{\bar{\alpha}\left(\varepsilon^{l}\right)+\sigma\Omega_{1}\left(\varepsilon^{l}\right)}>\frac{\bar{\alpha}\left(\varepsilon^{h}\right)+\sigma\Omega_{0}\left(\varepsilon^{h}\right)}{\bar{\alpha}\left(\varepsilon^{h}\right)+\sigma\Omega_{1}\left(\varepsilon^{h}\right)} \tag{30}$$

Putting together inequalities (29) and (30), we have

$$\begin{aligned}\frac{p}{1-p}&=\frac{\bar{\psi}_{0}\left(\varepsilon^{h}\right)}{1-\bar{\psi}_{1}(\varepsilon^{h})}\frac{\bar{\alpha}\left(\varepsilon^{h}\right)+\sigma\Omega_{0}\left(\varepsilon^{h}\right)}{\bar{\alpha}\left(\varepsilon^{h}\right)+\sigma\Omega_{1}\left(\varepsilon^{h}\right)}\\&<\frac{\bar{\psi}_{0}\left(\varepsilon^{l}\right)}{1-\bar{\psi}_{1}(\varepsilon^{l})}\frac{\bar{\alpha}\left(\varepsilon^{l}\right)+\sigma\Omega_{0}\left(\varepsilon^{l}\right)}{\bar{\alpha}\left(\varepsilon^{l}\right)+\sigma\Omega_{1}\left(\varepsilon^{l}\right)},\end{aligned}$$

which contradicts the equilibrium condition (28).

Now suppose instead that $\sigma<0$. In this case, the function $\psi$ is decreasing in $\delta$. As a result, we also have $\hat{\psi}\left(s,\bar{e},p\right)\geq\hat{\psi}(s,\underline{e},p)$ and $\hat{\psi}\left(s_{1},e,p\right)\leq\hat{\psi}\left(s_{0},e,p\right)$. The inequalities in (29) are thus reversed, and we have $\bar{\psi}_{\tau}\left(\varepsilon^{h}\right)\geq\bar{\psi}_{\tau}\left(\varepsilon^{l}\right)$. At the same time, the fact that aggregate exposure is increasing in $\varepsilon$ implies that (30) is reversed as well. But then

$$\begin{aligned}\frac{p}{1-p}&=\frac{\bar{\psi}_{0}\left(\varepsilon^{h}\right)}{1-\bar{\psi}_{1}(\varepsilon^{h})}\frac{\bar{\alpha}\left(\varepsilon^{h}\right)+\sigma\Omega_{0}\left(\varepsilon^{h}\right)}{\bar{\alpha}\left(\varepsilon^{h}\right)+\sigma\Omega_{1}\left(\varepsilon^{h}\right)}\\&>\frac{\bar{\psi}_{0}\left(\varepsilon^{l}\right)}{1-\bar{\psi}_{1}(\varepsilon^{l})}\frac{\bar{\alpha}\left(\varepsilon^{l}\right)+\sigma\Omega_{0}\left(\varepsilon^{l}\right)}{\bar{\alpha}\left(\varepsilon^{l}\right)+\sigma\Omega_{1}\left(\varepsilon^{l}\right)},\end{aligned}$$

which again contradicts the equilibrium condition (28).

Finally, consider now the case of exponential utility $(\sigma=0)$. The first order conditions for agent $\theta$ imply that

$$c_{0}\left(\theta\right)=c_{1}\left(\theta\right)+\alpha\left(\theta\right)\ln\left(\frac{p\left(1-\hat{\delta}\left(\theta,p\right)\right)}{\hat{\delta}\left(\theta,p\right)\left(1-p\right)}\right).$$

This equation, the individual budget constraints, and the market clearing condition for state $\tau=1$ imply that

$$\sum_{\theta}\mu_{j}\left(\theta;p\right)\left(\omega_{1}\left(\theta\right)-\omega_{0}\left(\theta\right)\right)=\sum_{\theta}\mu_{j}\left(\theta;p\right)\alpha\left(\theta\right)\left[\ln\left(\frac{\hat{\delta}\left(\theta,p\right)}{1-\hat{\delta}\left(\theta,p\right)}\right)-\ln\left(\frac{p}{1-p}\right)\right]\quad\text{for } j=l,h.$$

If $\sigma=0$, the exposure of a type $\theta$ agent is determined by the ratio

$$e_{1}\left(\theta\right)=\frac{\omega_{1}\left(\theta\right)-\omega_{0}\left(\theta\right)}{\alpha\left(\theta\right)}.$$

The previous two equations imply that

---

$$\frac{1}{1-\pi+\frac{1}{u'(\Omega_{0}(\varepsilon^{j}))/u'(\Omega_{1}(\varepsilon^{j}))-1}}=\frac{1}{1-\pi+\frac{1}{\left(\frac{\bar{\alpha}(\varepsilon^{j})+\sigma\Omega_{0}(\varepsilon^{j})}{\bar{\alpha}(\varepsilon^{j})+\sigma\Omega_{1}(\varepsilon^{j})}\right)^{-1/\sigma}-1}},$$

where $\pi$ denoted the probability attached to $\tau=1$.

$$\ln\left(\frac{p}{1-p}\right)=\sum_{\theta}\mu_j\left(\theta;p\right)\frac{\alpha\left(\theta\right)}{\bar{\alpha}\left(\varepsilon^j\right)}\varphi\left(\hat{\delta}\left(\theta,p\right),p\right)\quad\text{for } j=l,h, \tag{31}$$

where

$$\varphi\left(\delta,p\right)=\ln\left(\frac{\hat{\delta}\left(\theta,p\right)}{1-\hat{\delta}\left(\theta,p\right)}\right)-e_1\left(\theta\right).$$

Assume towards a contradiction that $\delta^h\left(p\right)\leq\delta^l\left(p\right)$. The function $\varphi$ is strictly increasing in $\delta$ and strictly decreasing in $e_1$, which implies $\varphi\left(s,\bar{e},p\right)<\varphi(s,\underline{e},p)$ and $\varphi\left(s_1,e,p\right)\geq\varphi\left(s_0,e,p\right)$. Now define $\tilde{\varepsilon}^j$ as

$$\tilde{\varepsilon}^j:=\frac{\varepsilon^j\alpha\left(\bar{e}\right)}{\varepsilon^j\alpha\left(\bar{e}\right)+\left(1-\varepsilon^j\right)\alpha\left(\underline{e}\right)}=\frac{1}{1+\frac{\left(1-\varepsilon^j\right)}{\varepsilon^j}\frac{\alpha\left(\underline{e}\right)}{\alpha\left(\bar{e}\right)}}.$$

For $\alpha\left(\underline{e}\right)$, $\alpha\left(\bar{e}\right)>0$, it is easy to verify that $\tilde{\varepsilon}^j$ is strictly increasing in $\varepsilon^j$.

Therefore,

$$\begin{aligned}\ln\left(\frac{p}{1-p}\right)&=\tilde{\varepsilon}^h\left[\delta^h\varphi\left(s_1,\bar{e},p\right)+\left(1-\delta^h\right)\varphi\left(s_0,\bar{e},p\right)\right]+\left(1-\tilde{\varepsilon}^h\right)\left[\delta^h\varphi\left(s_1,\underline{e},p\right)+\left(1-\delta^h\right)\varphi\left(s_0,\underline{e},p\right)\right]\\&<\tilde{\varepsilon}^l\left[\delta^h\varphi\left(s_1,\bar{e},p\right)+\left(1-\delta^h\right)\varphi\left(s_0,\bar{e},p\right)\right]+\left(1-\tilde{\varepsilon}^l\right)\left[\delta^h\varphi\left(s_1,\underline{e},p\right)+\left(1-\delta^h\right)\varphi\left(s_0,\underline{e},p\right)\right]\\&\leq\tilde{\varepsilon}^l\left[\delta^l\varphi\left(s_1,\bar{e},p\right)+\left(1-\delta^l\right)\varphi\left(s_0,\bar{e},p\right)\right]+\left(1-\tilde{\varepsilon}^l\right)\left[\delta^l\varphi\left(s_1,\underline{e},p\right)+\left(1-\delta^l\right)\varphi\left(s_0,\underline{e},p\right)\right]\\&=\ln\left(\frac{p}{1-p}\right),\end{aligned} \tag{32}$$

where the first inequality holds because $\varphi\left(s,\bar{e},p\right)<\varphi(s,\underline{e},p)$ and $\varepsilon^h>\varepsilon^l$, and where the second inequality holds because $\varphi\left(s_1,e,p\right)\geq\varphi\left(s_0,e,p\right)$ and $\delta^h\leq\delta^l$.

■

**Proof of Proposition 6.2.**

Using analogous notation as for the REE price function, we define $P_{FI}^j\left(\delta\right):=P_{FI}\left(\delta,\varepsilon^j\right)$. By (10) in the proof of Proposition 3.2, the functions $P_{FI}^j$ are strictly increasing and thus have well-defined inverse functions $\delta_{FI}^j\left(p\right)=\left(P_{FI}^j\right)^{-1}\left(p\right)$.

We now establish that for all $p\in(0,1)$,

$$\delta^h\left(p\right)>\delta_{FI}^h\left(p\right)>\delta_{FI}^l\left(p\right)>\delta^l\left(p\right).$$

Since both $\tilde{P}$ and $\tilde{P}_{FI}$ are strictly increasing and continuous in $\delta$, this proves part 3.

In the full information case, we can follow the same algebra as in the proof of Proposition 3.2. above to arrive at equation (28). If all beliefs are equal at $\hat{\delta}(\theta, p) = \delta^j(p)$, that equation simplifies to

$$\frac{p}{1-p} = \frac{\psi\left(\delta^j_{FI}(p), p\right)}{1-\psi\left(\delta^j_{FI}(p), p\right)} \frac{\bar{\alpha}\left(\varepsilon^j\right) + \sigma\Omega_0\left(\varepsilon^j\right)}{\bar{\alpha}\left(\varepsilon^j\right) + \sigma\Omega_1\left(\varepsilon^j\right)} \quad \text{for } j = h, l. \tag{33}$$

To establish $\delta^h(p) > \delta^h_{FI}(p)$, start again with the case $\sigma > 0$. Since all the $\bar{\psi}_\tau\left(\varepsilon^j\right)$ are averages of the $\psi\left(\hat{\delta}(\theta, p), p\right)$, and moreover the $\hat{\delta}(\theta, p)$ are averages of $\delta^h$ and $\delta^l$, we have $\bar{\psi}_\tau\left(\varepsilon^h\right) < \max_\theta \psi\left(\hat{\delta}(\theta, p), p\right) < \psi\left(\delta^h, p\right)$ for $\tau = 1, 2$. Therefore

$$\frac{p}{1-p} < \frac{\psi\left(\delta^h, p\right)}{1-\psi\left(\delta^h, p\right)} \frac{\bar{\alpha}\left(\varepsilon^h\right) + \sigma\Omega_0\left(\varepsilon^h\right)}{\bar{\alpha}\left(\varepsilon^h\right) + \sigma\Omega_1\left(\varepsilon^h\right)}. \tag{34}$$

The definition of $\delta^h_{FI}$ in (33) together with the fact that $\psi$ is strictly increasing thus implies $\hat{\delta}^h < \delta^h$. The argument for $\hat{\delta}^l > \delta^l$ follows analogously from the fact that $\bar{\delta}_\tau\left(\delta, \varepsilon^l\right) > \delta^l$ for $\tau = 1, 2$.

Now if $\sigma < 0$, the $\psi$s are decreasing in $\delta$, so that

$$\bar{\psi}_\tau\left(\varepsilon^h\right) > \min_\theta \psi\left(\hat{\delta}(\theta, p), p\right) > \psi\left(\delta^h, p\right)$$

and the inequality (34) is reversed. However, The definition of $\delta^h_{FI}$ in (33) together with the fact that $\psi$ is strictly decreasing once more implies $\hat{\delta}^h < \delta^h$. Again, the argument for $\hat{\delta}^l > \delta^l$ follows analogously from the fact that $\bar{\delta}_\tau\left(\delta, \varepsilon^l\right) > \delta^l$ for $\tau = 1, 2$.

case $\sigma = 0$**:**

We restrict attention to the case in which $\alpha > 0$ for all $\theta$ (agents are risk averse). In the full information case, if all beliefs are equal at $\hat{\delta}(\theta, p) = \delta^j(p)$, the equilibrium simplifies to

$$ln\left(\frac{p}{1-p}\right) = ln\left(\frac{\delta^j_{FI}(p)}{1-\delta^j_{FI}(p)}\right) - \frac{\Omega_1(\varepsilon^j) - \Omega_0(\varepsilon^j)}{\bar{\alpha}(\varepsilon^j)} \quad \text{for } j = h, l. \tag{35}$$

Since all the $\hat{\delta}(\theta, p)$ are averages of $\delta^h$ and $\delta^l$, we have $\hat{\delta}(\theta, p) < \delta^h$ for all $\theta$. The equilibrium condition (31) thus implies

$$\begin{aligned} \ln\left(\frac{p}{1-p}\right) &= \sum_\theta \mu_h(\theta; p) \frac{\alpha(\theta)}{\bar{\alpha}\left(\varepsilon^h\right)} ln\left(\frac{\hat{\delta}(\theta, p)}{1-\hat{\delta}(\theta, p)}\right) - \frac{\Omega_1(\varepsilon^h) - \Omega_0(\varepsilon^h)}{\bar{\alpha}(\varepsilon^h)} \\ &< ln\left(\frac{\delta^h(p)}{1-\delta^h(p)}\right) - \frac{\Omega_1(\varepsilon^h) - \Omega_0(\varepsilon^h)}{\bar{\alpha}(\varepsilon^h)} \quad \text{for all } p \in (0, 1) \end{aligned}$$

The definition of $\delta^h_{FI}$ in (35) thus implies $\hat{\delta}^h_{FI}(p) < \delta^h(p)$ for all $p \in (0,1)$. The argument for $\hat{\delta}^l_{FI}(p) > \delta^l(p)$ follows analogously from the fact that $\hat{\delta}(\theta, p) > \delta^l$ for all $\theta$.

■

## A.3 More General Preferences

This subsection provides an example that is minimal in terms of the shock structure—there are only two aggregate states, so $\delta$ and $\varepsilon$ are perfectly correlated—but makes no assumptions on preferences utility beyond expected utility. We show conditions for the existence of an equilibrium with asymmetric information and that the main property underlying our results—investors with more exposure to a risk factor are more optimistic about the risk factor in the presence of private information—is present also in this setting.

Nature draws the distribution of types $\mu^h = (\delta^h, \varepsilon^h)$ with probability $\eta$ and the distribution of types $\mu^l = (\delta^l, \varepsilon^l)$ with probability $1 - \eta$. We further assume that private signals $s(\theta)$ are fully uninformative about dividends, i.e., the distribution of $s(\theta)$ does not depend on $\delta$. For practical purposes, this means that there are two types of agents: $\Theta = \{\bar{\theta}, \underline{\theta}\}$, with type $\bar{\theta}$ having higher initial exposure than type $\underline{\theta}$.[18]

Since exposure is iid across agents, an individual agent's initial exposure is a signal about $\mu$. Pooling all agents' information about their own exposure reveals the distribution $\mu$, or, equivalently, the aggregate shock $(\delta, \varepsilon)$. To summarize, an economy is described by

$$\mathsf{E} = \left(\eta, \delta^h, \delta^l, \varepsilon^h, \varepsilon^l, \omega\left(\bar{\theta}\right), \omega\left(\underline{\theta}\right), u\left(.; \bar{\theta}\right), u\left(.; \underline{\theta}\right)\right).$$

Since the number of aggregates states $(\mu, \tau)$ that can occur is finite, prices in a rational-expectations equilibrium are fully revealing for a generic economy $\mathsf{E}$. To illustrate the effect of uncertain exposure on trading, we thus construct nongeneric economies that have nonrevealing equilibria. The economic mechanisms that emerge are also relevant in economies where $\Delta(\Theta)$ is uncountable and fully revealing equilibria do not exist. What is special about the present case is that, in a nonrevealing equilibrium, the price is constant across distributions $\mu^j$ and carries

[18] In this setup, the information set of an individual consists of his type (the price is not informative) and individual exposure is thus determined by

$$e\left(c; u(.;\theta); \theta\right) = \frac{1}{1 - \hat{\delta}(\theta) + \frac{1}{\frac{u'(c_0;\theta)}{u'(c_1;\theta)} - 1}}.$$

no information at all. Individual beliefs are thus independent of $\mu^j$ and depend only on agents' individual types; we write $\hat{\delta}(\theta)$ for the probability that type $\theta$ assigns to the event $\tau = \tau_1$. Individual consumption is also independent of $\mu$; we write $c_\tau(\theta)$, suppressing the dependence on $\mu$.

Proposition 6.3 constructs economies that have nonrevealing equilibria and characterizes their properties.

**Proposition 6.3**.

1. *An economy* $\mathsf{E}$ *can have at most one nonrevealing rational-expectations equilibrium. In such an equilibrium,*
   (a) $c(\theta) = \omega(\theta)$ *for all* $\theta$ *(there is no trade)*
   (b) $\hat{\delta}\left(\bar{\theta}\right) > \hat{\delta}\left(\underline{\theta}\right)$ *(agents with higher exposure to the risk factor* $\tau$ *are more optimistic about* $\tau$*: they find it more likely that* $\tau$ *takes on the high value* $\tau = 1$*)*
   (c) $\delta^h > \delta^l$ *(the aggregate news about the risk factor* $\tau$ *is better when there are more agents with higher exposure to the risk factor* $\tau$*)*

2. *The following conditions are equivalent:*
   (a) *there exist* $\delta^h, \delta^l \in (0,1)$ *such that the economy*
   $$\mathsf{E} = \left(\eta, \delta^h, \delta^l, \varepsilon^h, \varepsilon^l, \omega\left(\bar{\theta}\right), \omega\left(\underline{\theta}\right), u\left(.;\bar{\theta}\right), u\left(.;\underline{\theta}\right)\right)$$
   *has a nonrevealing rational-expectations equilibrium.*
   (b) *the endowments, felicities and distribution parameters* $\varepsilon$ *satisfy*
   $$\tilde{e}\left(\bar{\theta}\right) - \tilde{e}\left(\underline{\theta}\right) \le \log\left(\frac{\varepsilon^h}{1-\varepsilon^h}\frac{1-\varepsilon^l}{\varepsilon^l}\right).[19] \tag{36}$$

**Proof of Part 1.**

The proof of part 1 is straightforward. In a nonrevealing equilibrium, an agent of type $\theta$ must have the same beliefs in the two aggregate states $h$ and $l$. His net demand $c(\theta) - \omega\left(\bar{\theta}\right)$

[19] The term $\tilde{e}(\theta)$ is defined as

$$\tilde{e}(\theta) := \log\left(\frac{u'(\omega_0)}{u'(\omega_1)}\right) = \log\left(\frac{1 + \pi e\left(\omega(\theta), u(.;\theta), \varnothing\right)}{1 - (1-\pi)\, e\left(\omega(\theta), u(.;\theta), \varnothing\right)}\right).$$

must therefore be the same in both aggregate states. Market clearing requires that, for $j = h, l$ and $\tau = 1, 2$,

$$\varepsilon^j \left(c_\tau\left(\bar{\theta}\right) - \omega_\tau\left(\bar{\theta}\right)\right) + \left(1 - \varepsilon^j\right)\left(c_\tau\left(\underline{\theta}\right) - \omega_\tau\left(\underline{\theta}\right)\right) = 0.$$

For fixed $\tau$, this equation can only hold for both $j = h$ and $j = l$ if the net demands $c_\tau\left(\theta\right) - \omega_\tau\left(\theta\right)$ are zero for both types, which shows part 1.a.

Under our assumptions on felicities, agents' first-order conditions must hold in equilibrium. If agents consume their endowments, this means

$$\frac{p}{1-p} = \frac{\hat{\delta}\left(\theta\right)}{1 - \hat{\delta}\left(\theta\right)} \frac{u'\left(\omega_1\left(\theta\right), \theta\right)}{u'\left(\omega_0\left(\theta\right), \theta\right)} = \frac{\hat{\delta}\left(\theta\right)}{1 - \hat{\delta}\left(\theta\right)} \exp\left(-\tilde{e}\left(\theta\right)\right). \tag{37}$$

In an autarkic equilibrium, the type with higher initial exposure to the factor $\tau$ must be more optimistic that $\tau$ takes on the high value $\tau_1$, than the type with lower exposure, that is $\hat{\delta}\left(\bar{\theta}\right) > \hat{\delta}\left(\underline{\theta}\right)$ (part 1.b). Otherwise, agents with high and low initial exposure cannot both rationalize the observed price.

The existence of an equilibrium requires further that the beliefs in equation (37) can be derived by Bayesian updating from agents' individual types, which serve as noisy signals of the type distribution. In particular, an agent's subjective probability that $\tau = 1$ must be his conditional expectation of the aggregate news $\delta$, conditional on his type:

$$\begin{aligned} \hat{\delta}\left(\bar{\theta}\right) &= \eta \frac{\varepsilon^h}{\bar{\varepsilon}} \delta^h + \left(1 - \eta\right) \frac{\varepsilon^l}{\bar{\varepsilon}} \delta^l, \\ \hat{\delta}\left(\underline{\theta}\right) &= \eta \frac{1 - \varepsilon^h}{1 - \bar{\varepsilon}} \delta^h + \left(1 - \eta\right) \frac{1 - \varepsilon^l}{1 - \bar{\varepsilon}} \delta^l, \end{aligned} \tag{38}$$

where we have defined $\bar{\varepsilon} := \eta \varepsilon^h + \left(1 - \eta\right) \varepsilon^l$, the unconditional probability of type $\bar{\theta}$. Since $\varepsilon^h > \varepsilon^l$ and $\hat{\delta}\left(\bar{\theta}\right) > \hat{\delta}\left(\underline{\theta}\right)$, we must have $\delta^h > \delta^l$ (part 1.c). Moreover, given a pair of $\delta^j$s in $(0, 1)$ and hence an economy, the formulas (38) deliver a unique pair of posteriors, and (37) the unique nonrevealing equilibrium price.

Intuitively, existence of a nonrevealing equilibrium requires that the probability $\delta$ is higher when more agents have high exposure to the factor $\tau$. High exposure agents then interpret their type as a signal that $\tau = 1$. Since they are more optimistic about $\tau$, they are happy to consume their endowment at the price $p$, even though they have higher exposure than other agents (so that gains from trade would exist with symmetric information).

**Proof of Part 2.**

The proof of part 2 starts from the fact that for given $\hat{\delta}(\theta)$s, (38) can be viewed as a pair of linear equations in $\left(\delta^h, \delta^l\right)$ with a unique solution. The existence problem then amounts to finding a price such that, if the posteriors satisfy (37), then the solutions $\left(\delta^h, \delta^l\right)$ to (38) are indeed between zero and one. Such a price exists if and only if condition (36) is satisfied. The condition requires that requires that there should not be "too much" heterogeneity in individual exposure, relative to the differences in type distributions across states.

We first need to establish the existence of $\left(\delta^h, \delta^l\right)$ and a price $p$ such that a price function that is constant at $p$ together with the autarkic allocation constitute an equilibrium in the economy parameterized by the $\delta$s.

Given our assumptions on utility, it is optimal for agents to consume their endowment at the price $p$ and for belief $\hat{\delta}(\theta)$ if and only if the first order conditions

$$\frac{\hat{\delta}(\theta)}{1-\hat{\delta}(\theta)} \exp\left(-e_1(\theta)\right) = \frac{p}{1-p}.$$

hold for every $\theta$. In other words, equilibrium posteriors must be

$$\hat{\delta}(\theta) = \left(1 + \frac{1-p}{p} \exp\left(-e_1(\theta)\right)\right)^{-1}. \tag{39}$$

We are done if we can show that there exist $\left(\delta^h, \delta^l\right)$ and $p$ such that the posteriors $\hat{\delta}(\theta)$ not only satisfy (39), but are also derived from agents' individual types by Bayes' Rule. If this is true, then the $\hat{\delta}(\theta)$ are also posteriors given a constant, and hence uninformative, price function. (39) thus says that the autarkic allocation is optimal in every state given the price $p$. Finally, markets clear in all states if each consumer chooses his endowment.

Consider agents' updating given their individual type. Bayes' Rule says

$$\hat{\delta}\left(\bar{\theta}\right) = \frac{\eta\varepsilon^h\delta^h + (1-\eta)\,\varepsilon^l\delta^l}{\eta\varepsilon^h + (1-\eta)\,\varepsilon^l} = \eta\frac{\varepsilon^h}{\bar{\varepsilon}}\delta^h + (1-\eta)\,\frac{\varepsilon^l}{\bar{\varepsilon}}\delta^l,$$

$$\hat{\delta}\left(\underline{\theta}\right) = \frac{\eta\left(1-\varepsilon^h\right)\delta^h + (1-\eta)\left(1-\varepsilon^l\right)\delta^l}{\eta\left(1-\varepsilon^h\right) + (1-\eta)\left(1-\varepsilon^l\right)} = \eta\frac{1-\varepsilon^h}{1-\bar{\varepsilon}}\delta^h + (1-\eta)\,\frac{1-\varepsilon^l}{1-\bar{\varepsilon}}\delta^l,$$

where we have defined $\bar{\varepsilon} = \eta\varepsilon^h + (1-\eta)\,\varepsilon^l$.

For fixed $p$, this can be viewed as a linear equation in $\left(\delta^h, \delta^l\right)$ with unique solution

$$\delta^h = \frac{\left(1-\varepsilon^l\right)\bar{\varepsilon}\hat{\delta}\left(\bar{\theta}\right) - \varepsilon^l\left(1-\bar{\varepsilon}\right)\hat{\delta}\left(\underline{\theta}\right)}{\eta\left(\varepsilon^h - \varepsilon^l\right)},$$

$$\delta^l = \frac{\varepsilon^h\left(1-\bar{\varepsilon}\right)\hat{\delta}\left(\underline{\theta}\right) - \left(1-\varepsilon^h\right)\bar{\varepsilon}\hat{\delta}\left(\bar{\theta}\right)}{(1-\eta)\left(\varepsilon^h - \varepsilon^l\right)}. \tag{40}$$

We must ensure that $\delta^h$ and $\delta^l$ are between zero and one. The inequalities $\delta^h > 0$ and $\delta^l > 0$ are equivalent to the two inequalities in

$$\frac{\varepsilon^l}{1-\varepsilon^l} < \frac{\bar{\varepsilon}}{1-\bar{\varepsilon}} \frac{\hat{\delta}\left(\bar{\theta}\right)}{\hat{\delta}\left(\underline{\theta}\right)} < \frac{\varepsilon^h}{1-\varepsilon^h}, \tag{41}$$

respectively. Moreover, the inequalities $\delta^h < 1$ and $\delta^l < 1$ are equivalent to the inequalities in

$$\frac{\varepsilon^l}{1-\varepsilon^l} < \frac{\bar{\varepsilon}}{1-\bar{\varepsilon}} \frac{1-\hat{\delta}\left(\bar{\theta}\right)}{1-\hat{\delta}\left(\underline{\theta}\right)} < \frac{\varepsilon^h}{1-\varepsilon^h}, \tag{42}$$

respectively.

To simplify notation, we write $\bar{\rho} = \exp\left(e_1\left(\bar{\theta}\right)\right)$ and $\underline{\rho}$=$\exp\left(e_1\left(\underline{\theta}\right)\right)$. From agents' first order conditions, we know

$$\frac{\hat{\delta}\left(\bar{\theta}\right)}{\hat{\delta}\left(\underline{\theta}\right)} = \frac{p+(1-p)/\underline{\rho}}{p+(1-p)/\bar{\rho}} =: f\left(p\right)$$

$$\frac{1-\hat{\delta}\left(\bar{\theta}\right)}{1-\hat{\delta}\left(\underline{\theta}\right)} = \frac{1-p+p\underline{\rho}}{1-p+p\bar{\rho}} =: g\left(p\right)$$

We want to show that there exists a price $p \in (0,1)$ such that

$$f\left(p\right), g\left(p\right) \in \left[\frac{\varepsilon^l}{1-\varepsilon^l}\frac{1-\bar{\varepsilon}}{\bar{\varepsilon}}, \frac{\varepsilon^h}{1-\varepsilon^h}\frac{1-\bar{\varepsilon}}{\bar{\varepsilon}}\right] =: \left[\underline{b}, \bar{b}\right]$$

If such a price exists, then the $\delta$s in (40) are between zero and one, and therefore $p$ is a nonrevealing equilibrium price for the economy parameterized by those $\delta$s. By construction, we have $\bar{b} > 1 > \underline{b}$. This already shows that there exists an equilibrium price if the differences in exposure are not "too large": if $\bar{\rho} = \underline{\rho}$, then $f\left(p\right) = g\left(p\right) = 1$ for any price. By continuity, an equilibrium also exists for "small enough" heterogeneity. We now establish that condition (36) provides tight bounds for this heterogeneity.

Using the fact that $\bar{\rho} > 1$ and $\bar{\rho} > \underline{\rho}$, it can be verified that the function $f$ is continuous and strictly decreasing for all $p > p_f$, where

$$p_f = -\frac{\bar{\rho}}{\bar{\rho}-1}$$

Furthermore $f\left(0\right) = \bar{\rho}/\underline{\rho} > 1$ and $f\left(1\right) = 1$ and $f$ tends to $+\infty$ as $p$ tends to $p_f$ from above. It follows that $f\left(p\right) \geq \underline{b}$ for all $p \in (0,1)$. and that there exists a unique price $p^u > p_f$ such that $f\left(p^u\right) \leq \bar{b}$ for all $p \geq p^u$.

It can also be verified that the function $g$ is continuous and strictly decreasing for all $p > p_g$, where

$$p_g = -\frac{1}{\bar{\rho}-1} > p_f.$$

Furthermore $g(0) = 1$ and $g(1) = \underline{\rho}/\bar{\rho} < 1$ and $g$ tends to $+\infty$ as $p$ tends to $p_g$ from above. It follows that $g(p) \leq \bar{b}$ for all $p \in (0,1)$. We also know that $f(p) > g(p)$ for all $p \in (0,1)$.

It follows that there exists a price in $p \in (0,1)$ such that $f(p), g(p) \in [\underline{b}, \bar{b}]$ if and only if $g(p^u) \geq \underline{b}$. Indeed, suppose that $g(p^u) \geq \underline{b}$. Since $f(1) = 1$, we know that $p^u < 1$. If $p^u \in (0,1)$, then $f(p^u), g(p^u) \in [\underline{b}, \bar{b}]$. If $p^u < 0$, then $f(0) < \bar{b}$. But we also have $f(0) > g(0) = 1 > \underline{b}$. Using continuity of $f$ and $g$, we can therefore pick a small positive price $p$ such that $f(p), g(p) \in [\underline{b}, \bar{b}]$. To show the converse, suppose that $g(p^u) < \underline{b}$. Since $g(0) = 1$, it must be that $p^u > 0$. Since $g$ is decreasing, we have $g(p) < \underline{b}$ for all $p \in [p^u, 1)$. But at the same time, $f(p) > \bar{b}$ for all $p \in (0, p^u)$. As a result there exists no price in the unit interval such that $f(p), g(p) \in [\underline{b}, \bar{b}]$.

We now show that the condition $g(p^u) \geq \underline{b}$ is equivalent to condition (36). We first solve for $p^u$ from the equation $f(p^u) = \bar{b}$ to find

$$\frac{p^u}{1-p^u} = \frac{\underline{\rho}^{-1} - \bar{b}\bar{\rho}^{-1}}{\bar{b}-1}$$

The condition $g(p^u) \geq \underline{b}$ is

$$\frac{\frac{1-p^u}{p^u} + \underline{\rho}}{\frac{1-p^u}{p^u} + \bar{\rho}} \geq \underline{b}.$$

Substituting in for $\frac{p^u}{1-p^u}$ and multiplying the numerator and denominator by $\underline{\rho}^{-1} - \bar{b}\bar{\rho}^{-1}$, we obtain equivalently

$$\frac{\left(\bar{b}-1\right) + \underline{\rho}\left(\underline{\rho}^{-1} - \bar{b}\bar{\rho}^{-1}\right)}{\left(\bar{b}-1\right) + \bar{\rho}\left(\underline{\rho}^{-1} - \bar{b}\bar{\rho}^{-1}\right)} \geq \underline{b},$$

which simplifies to

$$\frac{\bar{b}\left(1 - \underline{\rho}/\bar{\rho}\right)}{\bar{\rho}/\underline{\rho} - 1} \geq \underline{b}.$$

and further to

$$\bar{\rho}/\underline{\rho} \leq \bar{b}/\underline{b}$$

Using the definitions of $\bar{\rho}$, $\underline{\rho}$, $\bar{b}$ and $\underline{b}$ we arrive at the condition (36).■

## A.4 Proof of Proposition 6.4

**Proof.** The agent solves

$$\max_{c \in C} -E[\exp(-\rho c) \,|I(\theta)]$$
$$s.t. \;\; P(c) = P(\omega).$$

Using the fact that a consumption plan can be represented as $c = a_c + b_c\tau$, and using the properties of normal distributions, the agent's problem simplifies to a linear quadratic problem in the coefficients:

$$\max_{a_c,b_c} \ \{\rho a_c + \rho b_c E[\tau|I(\theta)] - \frac{1}{2}\rho^2 b_c^2 var(\tau|\theta)\}$$
$$s.t.\ \ a_c + b_c p = a_\omega(\theta) + b_\omega(\theta)p$$

The coefficients of the optimal consumption bundle are then

$$b_c(\theta) = \frac{E[\tau|I(\theta)] - p}{\rho var(\tau|I(\theta))}$$
$$a_c(\theta) = a_\omega(\theta) + b_\omega(\theta)p - p\frac{E[\tau|I(\theta)] - p}{\rho var(\tau|I(\theta))}$$

The agent will load more on the factor if the expected excess return on the factor is higher, and when risk and risk aversion is lower. The endowment does not matter for the loading on the factor, or the agent's choice of risky assets. However, riskless claims are chosen so as to satisfy the budget constraint.

*Equilibrium*

Market clearing requires $\int c(\theta)\,d\theta = \int \omega(\theta)\,d\theta$, or, in terms of coefficients,

$$\int a_c(\theta)\,d\theta = \int a_\omega(\theta)\,d\theta$$
$$\int b_c(\theta)\,d\theta = \int b_\omega(\theta)\,d\theta$$

The first equation can be thought of as market clearing for riskless claims, and the second equation as market clearing for claims with payoff $\tau$. Walras' law holds for these two assets: if one equation holds, and the budget constraints, too, then the other market clears as well.

1. We prove the first claim by conjecturing on a linear price function

$$\tilde{P}(\delta,\varepsilon) = \bar{p} + \beta\delta + \gamma\varepsilon.$$

The projection theorem implies that the posterior mean $E[\delta \mid I(\theta)]$ of type $\theta$ can be written as

$$E[\delta|I(\theta)] = \frac{1}{\pi_\delta^*}\left(\pi_u s(\theta) - \frac{\beta}{\gamma}\pi_v b_\omega(\theta) + \frac{\beta}{\gamma^2}(\pi_v + \pi_\varepsilon)p\right), \tag{43}$$

where $\pi_\delta^*$ denotes the inverse of investor $\theta$'s posterior variance about the aggregate news $\delta$, i.e.,

$$(\pi_\delta^*)^{-1} := var(\delta | I(\theta)) = \left(\pi_\delta + \pi_u + \left(\frac{\beta}{\gamma}\right)^2 (\pi_v + \pi_\varepsilon)\right)^{-1}. \tag{44}$$

Since there is no information about $w$, $E[\tau \mid I(\theta)] = E[\delta \mid I(\theta)]$ and $V[\tau \mid I(\theta)] = \pi_\delta^{*-1} + \pi_w^{-1}$. After substituting the posterior moments for $\tau$ into individual demand equations, the market clearing condition implies that the equilibrium price must satisfy the following expression.

$$p = \frac{\pi_\delta^* - \pi_\delta}{\pi_\delta^*}\delta + \left[\frac{\beta}{\gamma}\frac{\pi_\varepsilon}{\pi_\delta^*} - \rho(\pi_\delta^{*-1} + \pi_w{}^{-1})\right]\varepsilon \quad \text{for all } \delta, \epsilon.$$

There exists a linear price function as long as the coefficients $\beta$ and $\gamma$ satisfy the non-linear equations

$$\beta = \frac{\pi_\delta^* - \pi_\delta}{\pi_\delta^*} \tag{45}$$

$$\gamma = \frac{\beta}{\gamma}\frac{\pi_\varepsilon}{\pi_\delta^*} - \rho(\pi_\delta^{*-1} + \pi_w{}^{-1}) \tag{46}$$

Dividing equation (45) by (46) we obtain

$$\beta/\gamma = \frac{(\beta/\gamma)^2(\pi_v + \pi_\varepsilon) + \pi_u}{(\beta/\gamma)\pi_\varepsilon - (\beta/\gamma)^2\rho\pi_w^{-1}(\pi_v + \pi_\varepsilon)) - \rho(1 + \pi_w^{-1}(\pi_u + \pi_\delta))}$$

It follows that the equilibrium values of $\beta/\gamma$ are determined by the roots of the cubic polynomial

$$g(x) = x^3\rho\pi_w^{-1}(\pi_v + \pi_\varepsilon) + x^2\pi_v + x\rho(1 + \pi_w^{-1}(\pi_u + \pi_\delta)) + \pi_u. \tag{47}$$

We have $\lim_{x\to-\infty} g(x) = -\infty$, $g(0) = \pi_u > 0$ and $g(x) > 0$ for all $x > 0$. By continuity of $g$, there is always at least one $\tilde{x} < 0$ with $g(\tilde{x}) = 0$. Moreover, all roots of $g$ satisfy $\tilde{x} < 0$. This proves part 1.

2. A zero of $g$ corresponds to an equilibrium with

$$\beta = \frac{\tilde{x}^2(\pi_v + \pi_\varepsilon) + \pi_u}{\tilde{x}^2(\pi_v + \pi_\varepsilon) + \pi_u + \pi_\delta} \in (0,1)$$
$$\gamma = \frac{\tilde{x}\pi_\varepsilon - \rho}{\tilde{x}^2(\pi_v + \pi_\varepsilon) + \pi_u + \pi_\delta} - \rho/\pi_w < 0$$

3. It is easy to verify that in the pooled information case

$$\tilde{P}^P(\delta, \varepsilon) = \beta^P\delta + \gamma^P\varepsilon = \delta - \rho/\pi_w\varepsilon,$$

which implies

$$\beta < \beta^P = 1, \quad \text{and}$$
$$\gamma < \gamma^P = -\rho/\pi_w,$$

Moreover, we have $\beta/\gamma > \beta^P/\gamma^P$.

4. It follows from $\beta/\gamma < 0$. ■